Core Course – 9

Elements of Modern Physics

Problem Set – 2 (Unit 2)

1. Normalize the following wave functions.

(i) $\psi(x) = A\cos x$ for $|x| \le \frac{\pi}{2}$
$= 0$ otherwise.

(ii) $\psi(x) = A\sin\frac{n\pi x}{l}$ for $-\frac{l}{2} \le x \le \frac{l}{2}$
$= 0$ otherwise.

(iii) $\psi(x) = A\,exp(-\alpha x^2)$ for $-\infty \le x \le \infty$

(iv) $\psi(r) = A\,exp(-r/a_0)$ for $0 \le r \le \infty$

(v) $\psi(\phi) = A\,exp(im\phi)$ for $0 \le \phi \le 2\pi$

2. The wave function of a particle is described as $\psi(x) = \sqrt{2}x$ for $0 < x < 1$ and $\Psi(x) = 0$ elsewhere. Find the probability of finding the particle in the regions $0 < x < 0.5$ and $0.5 < x < 1$. What is the average value of its position?

3. Given $\hat{A}$ and $\hat{B}$ are two linear operators. Show that their sum and product i.e. $\hat{A} + \hat{B}$ and $\hat{A}\hat{B}$ are also linear.

4. Check whether the following operators are linear.

(i) $\hat{A} = \frac{d}{dx}$ (ii) $\hat{A}f(x) = f^*(x)$ (iii) $\hat{A}f(x) = -f(x)$

(iv) Momentum operator represented by $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

(v) Hamiltonian operator represented by $\widehat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

5. Verify the following identities related to the commutator of operators.

(i) $[\hat{B}, \hat{A}] = -[\hat{A}, \hat{B}]$

(ii) $[\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}]$

(iii) $[\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}]$

(iv) $\left[\hat{A}, [\hat{B}, \hat{C}]\right] + \left[\hat{B}, [\hat{C}, \hat{A}]\right] + \left[\hat{C}, [\hat{A}, \hat{B}]\right] = 0$

6. Evaluate the following commutation brackets.

(i) $\left[\hat{x}, \frac{\partial}{\partial x}\right]$ (ii) $[\hat{x}, \hat{p}_x]$ (iii) $[\widehat{x^2}, \hat{p}_x]$ (iv) $[\widehat{x^n}, \hat{p}_x]$ (v) $\left[\hat{x}, \widehat{p_x^2}\right]$ (vi) $[\hat{x}, \widehat{p_x^n}]$

(vii) $[\hat{p}_x, \hat{p}_y]$ (viii) $[\widehat{H}, \hat{p}_x]$ (ix) $[\hat{p}_x, e^{\hat{x}}]$ (x) $[\hat{x}, e^{\widehat{p_x}}]$ (xi) $[\hat{x}, \sin\hat{p}_x]$

(xii) $[\widehat{L_x}, \widehat{L_y}]$ (xiii) $[\widehat{L_y}, \widehat{L_z}]$ (xiv) $[\widehat{L_z}, \widehat{L_x}]$ (xv) $[\widehat{L^2}, \widehat{L_x}]$ (xvi) $[\widehat{L^2}, \widehat{L_y}]$

(xvii) $[\widehat{L^2}, \widehat{L_z}]$

7. Check whether the following operators are Hermitian.

(i) Position operator: $\hat{x}$

(ii) Momentum operator: $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

(iii) $\hat{A} = (\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$

(iv) $\hat{B} = (\hat{x}\hat{p}_x - \hat{p}_x\hat{x})$

(v) Hamiltonian operator: $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)$

(vi) The third component of angular momentum: $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

8. Given, operators $\hat{A}$ and $\hat{B}$ both are Hermitian.

(i) Prove that $\hat{A} + \hat{B}$ is also Hermitian.

(ii) $\hat{A}\hat{B}$ is Hermitian if and only if the operators commute.

(iii) Is the commutator i.e. $[\hat{A}, \hat{B}]$ Hermitian?

9. Check whether the following wave functions are well-behaved?

(i) $\psi(x) = \exp(-\alpha x^2), \alpha > 0$

(ii) $\psi(x) = \exp(\alpha x^2), \alpha < 0$

(iii) $\psi(x) = \exp(kx), k > 0$

(iv) $\psi(x) = \exp(-kx), k > 0$

(v) $\psi(x) = \exp(ikx), k$ real

(vi) $\psi(x) = \cos kx, k$ real

(vii) $\psi(x) = Ax\exp(x^2)$ [$-\infty < x < \infty$ in all cases]

10. Find eigenfunction of the following operators:

(i) Momentum operator represented by $\hat{p}_x = -i\hbar\frac{\partial}{\partial x}$

(ii) Free particle Hamiltonian operator described by $\hat{H} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$

(iii) The third component of angular momentum: $\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$

11. Show that $\psi(x) = \exp(ikx)$ is an eigenfunction of the momentum operator. What is the corresponding eigenvalue? What is the expectation value of the momentum in this state? Is $\psi(x)$ simultaneous eigenfunction of the Hamiltonian operator for a free particle? If yes, find corresponding energy eigenvalue.

12. Show that the eigenvalues of Hermitian operators are real and the eigenfunctions corresponding to different eigenvalues are orthogonal.

13. Consider a wave function given by the following Gaussian distribution

$$\psi(x) = A\, e^{-\alpha(x-a)^2},$$

where A, a and α are positive real constants.

(i) Find out the normalization constant A.

(ii) Find the probability current density.

(iii) Find $\langle x\rangle, \langle x^2\rangle$ and hence the uncertainty in determination of x i.e. Δx.

14. Consider the wave function

$$\psi(x,t) = A\, e^{-\alpha|x|}e^{-i\omega t},$$

where A, a and α are positive real constants.

(i) Find out the normalization constant A.

(ii) Find $\langle x \rangle, \langle x^2 \rangle$ and hence the standard deviation of x.

15. Consider the following wave function

$$\psi(x) = A\, e^{-\alpha x^2},$$

where A and α are positive real constants.

(i) Find out the normalization constant A.

(ii) Calculate the expectation values of x, x^2, p and p^2.

(iii) Find Δx and Δp and show that their product is consistent with the uncertainty principle.

16. The wave function for a particle confined in a one dimensional potential box ($0 \leq x \leq l$) is given by

$$\psi(x) = A\ \sin\frac{n\pi x}{l}, \qquad n = 1, 2, 3, \dots \dots$$

(i) Find out the normalization constant A.

(ii) What is the probability that the particle can be found in the region $0 < x < l/2$.

(iii) Calculate the expectation values of x, x^2, p and p^2.

(iv) Find Δx and Δp and show that their product is consistent with the uncertainty principle.

(v) Is this wave function an eigenfunction of the corresponding Hamiltonian operator? If yes, find the energy eigenvalues.

17. Consider the following wave function in radial coordinate (a is a positive constant)

$$\psi(r) = A\, e^{-r/a}$$

(i) Calculate the normalization constant A.

(ii) Find the expectation value of the radial position r.

(iii) What is the probability that the system can be found at $0 < r < a$?

18. Show that Schrödinger's equation is linear. (Or, Given ψ_1 and ψ_2 are two solutions of Schrödinger's equation. Show that the linear combination of ψ_1 and ψ_2 will the general solution of Schrödinger's equation.)

19. A free particle wave function is given by $\psi(x,t) = A \exp i(kx - \omega t)$, A constant. Find the probability current density and show that wave function is consistent with the equation of continuity.

20. Consider a wave function $\psi(r) = e^{ikr}/r$, symbols have their usual meanings. Find the probability current density.

21. A particle in a stationary state with energy E_0 is at ψ_0 at time $t = 0$. After how much time will the particle again come back to the same state is at ψ_0?

22. The normalized wave function of a particle is given by $\psi = \frac{1}{\sqrt{3}}\psi_1 + \frac{2}{\sqrt{3}}\psi_2$ where ψ_1 and ψ_2 represent two orthonormal eigenstates of the Hamiltonian operator with energy eigenvalues E and $2E$

respectively. Find the probabilities that the particle is in the states ψ_1 and ψ_2. Also find the expectation value of the energy when it is in the state ψ.

23. A particle has two eigenstates ψ_1 and ψ_2 with the corresponding energy being E_1 and E_2 respectively. The particle has respectively 40% and 60% probabilities of being found in the states ψ_1 and ψ_2. Write down the wave function of the particle and obtain the average energy of the particle.

24. A system has two possible energy eigenvalues E_0 and $2E_0$ with corresponding eigenstates ψ_0 and ψ_1 respectively. The particle, at an instant, is in a state ψ such that it has an expectation value of energy $\frac{3}{2}E_0$. Find ψ. What is the value of ψ after a time t has elapsed?

25. The expectation value of an operator A is defined as $\langle A\rangle = \langle\psi|A|\psi\rangle$. Show that ($H$ is Hamiltonian operator)

$$\frac{d}{dt}\langle A\rangle = \langle\frac{\partial A}{\partial t}\rangle + (i\hbar)^{-1}\langle[A,H]\rangle$$

26. Using the relation in Prob. 25 or otherwise, prove the following relations (Ehrenfest Theorems)

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p_x\rangle}{m}$$

$$\frac{d}{dt}\langle p_x\rangle = -\langle\frac{\partial V}{\partial x}\rangle$$

Review of CU Exam. Papers:

CU – 2019

1. Find the eigenstate of $i\frac{d}{dx}$. [2]

2. Show that the eigenvalues corresponding to Hermitian operators are real. [2]

3. $\hat{A}$ and $\hat{B}$ are two Hermitian operators. Comment on the hermiticity of the commutator $[\hat{A},\hat{B}]$. [2]

4. Consider 1D harmonic oscillator potential $V(x) = m\omega^2x^2/2$. The ground state wave function is given by

$$\psi(x) = \left(\frac{\alpha}{\pi}\right)^{1/4} e^{-\alpha x^2/2}, \text{where } \alpha = \frac{m\omega}{\hbar}.$$

Calculate the expectation value of kinetic energy and potential energy and hence show that ground state energy is given by $E_0 = \hbar\omega/2$. [2+2+1]

5. For any operator $\hat{A}$ with no explicit time dependence, prove that

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{A},\hat{H}]\rangle$$

Hence prove that

$$\frac{d}{dt}\langle\hat{p}_x\rangle = -\langle\frac{\partial V(\hat{x})}{\partial x}\rangle$$

where $\hat{p}_x$ is the linear momentum, and the potential is $V(\hat{x})$. [3+2]

6. What is the implication of the relation $[\hat{A}, \hat{B}] = 0$, where $\hat{A}$ and $\hat{B}$ are two quantum mechanical observables? [2]

7. The Hamiltonian of a 1D system is given by

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + V(\hat{x})$$

Show that $[\hat{x}, [\hat{x}, \hat{H}]] = -\hbar^2/m$. [3]

8. If $[\hat{A}, \hat{B}] = 0$ and ψ is an eigenvector of $\hat{A}$ with eigenvalue λ, then show that $\hat{B}\psi$ is also an eigenvector of $\hat{A}$ with the same eigenvalue λ. Will ψ be an eigenvector of $\hat{B}$? [2+1]

9. Examine whether the operator $\hat{B}$ is linear or not, where $\hat{B}\psi(x) = \psi^*(x)$. [2]

CU – 2018

1. $\psi_1(x)$ and $\psi_2(x)$ are eigenstates of the Hamiltonian with eigenvalues E_1 and E_2. Is

$$\psi(x,t) = c_1\psi_1(x)e^{-iE_1t/\hbar} + c_2\psi_2(x)e^{-iE_2t/\hbar}$$

a stationary state? [2]

2. What is the physical significance of the fact that an operator commutes with Hamiltonian operator? [2]

3. The wave function of particle bound inside a one dimensional box is given by $\psi(x) = A\cos kx + B\sin kx$. If the two ends of the box are at $x = -a$ and $x = +a$ respectively, then write down the appropriate boundary conditions and modify the wave function accordingly. [2]

4. What do you mean by square integrable wave function? Cite an example of a wave function which is not square integrable. [2]

5. A particle is represented by

$$\psi(x) = A(a^2 - x^2), -a \leq x \leq a$$
$$= 0, \text{otherwise}$$

where A is the normalization constant. Determine the uncertainty in the position of the particle in terms of A and a. [4]

6. A system has two possible energy values E_0 and $2E_0$ and at a certain instant, the system is in a state in which the expectation value of energy is $3E_0/2$. Calculate the wave function in this state, given that ψ_0 and ψ_1 are the wave functions corresponding to the two possible energy values E_0 and $2E_0$ respectively. What is the wave function after a time t has elapsed? Assume that the eigenvalues E_0 and $2E_0$ are non-degenerate and the relative phase between the states ψ_0 and ψ_1 is θ. [4]

7. For a free particle, show that each energy eigenvalue is doubly degenerate. [2]

8. Show that the momentum operator is a Hermitian operator. [3]

9. Show that if H is Hermitian then e^{iH} is unitary.

10. Show that any operator A which has no explicit time dependence, follows

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle [A, H]\rangle$$

CU – 2017

1. Starting from the basic commutation relation $[x, p_x] = i\hbar$, one can show that $[x, p_x^n] = i\hbar n p_x^{n-1}$. Using this result or in other way, prove that $[\hat{x},\ \sin\hat{p}_x] = i\hbar\cos\hat{p}_x$. [2]

2. Write down the time independent Schrodinger equation for a particle moving in 1 – D. If there is no force acting on it, find out the solution. If the particle is constrained to move such that $0 < x < L$, what are the boundary conditions of the wave function? [3]

3. Using the commutation relation for the components of the momentum and position operators, show that the components of the angular momentum operator L satisfy $[L_x, L_y] = i\hbar L_z$. [4]

4. Show that for an operator A with no explicit time dependence

$$\frac{d}{dt}\langle A\rangle = \frac{i}{\hbar}\langle [A, H]\rangle$$

Hence, prove that

$$\frac{d}{dt}\langle p_x\rangle = -\langle\frac{\partial V}{\partial x}\rangle$$

for a particle moving in x direction with momentum p_x under the potential $V(x)$. [3+3]

5. Prove Ehrenfest theorem in one dimension [4]

$$\frac{d}{dt}\langle x\rangle = \frac{\langle p_x\rangle}{m}$$

CU – 2016

1. A particle at $t = 0$ is described by wave function

$$\Psi(x) = A\, e^{-\alpha x^2} e^{i k_0 x}$$

Find $\langle x\rangle$. [3]

2. A particle can be in two different states given by orthonormal wave functions ψ_1 and ψ_2. If the probability of being in state ψ_1 is $1/3$, find out the normalized wave function for the particle. [2]

3. Find the explicit expression of the operator $\left(A\frac{d}{dA}\right)^2$. [3]

4. Stating from the basic commutation relation $[x, p_x] = i\hbar$, prove that $[x, p_x^n] = i\hbar n p_x^{n-1}$. [2]

5. Show that the momentum operator $p_x = -i\hbar\frac{\partial}{\partial x}$ is Hermitian in nature. [4]

6. What is the physical significance of $[x, p_y] = 0$? [1]

CU – 2015

1. Show that if ψ be an eigen function of the operator $\hat{A}$ with eigenvalue λ, then it is also eigenfunction of e^{A} with eigen value e^{λ}. [2]

2. Examine whether the operator $\hat{B}$ is linear or not, where $\hat{B}\psi(x) = \psi^*(x)$. [2]

3. Show that the operator $\hat{Q} = \frac{1}{2}(\hat{x}\hat{p}_x + \hat{p}_x\hat{x})$ is Hermitian. [3]

4. Show that if F is Hermitian, $U = e^{iF}$ is Hermitian. [1]

5. What is the momentum representation of the position operator $\hat{x}$? Show that this representation satisfies the position-momentum commutation relation. [2+2]

6. For any operator $\hat{A}$ which has no explicit time dependence, follows

$$\frac{d}{dt}\langle\hat{A}\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{A}]\rangle_t$$

hence prove that

$$\frac{d}{dt}\langle P_x\rangle = -\langle\frac{dV(x)}{dx}\rangle$$

for any particle moving in x direction with momentum P and under the potential $V(x)$. [3]

CU – 2014

1. Calculate the normalization constant and probability current density for a wave function given by (at $t = 0$) [3]

$$\psi(x) = A\, e^{-\alpha x^2/2} e^{ikx}$$

2. Prove that $\exp[i(AB - BA)]$ is a Hermitian operator if A and B are Hermitian operators. [3]

3. Prove the relation $[A, BC] = B[A, C] + [A, B]C$ and using this evaluate $[p_x, x^2]$. [2+2]

CU – 2013

1. Let the state of a system be denoted by $|\psi\rangle = a_1|\phi_1\rangle + a_2|\phi_2\rangle$. If the probability that the system is in the state $|\phi_1\rangle$ is double of that in $|\phi_2\rangle$, find out a_1 and a_2. [2]

2. If the operators $\hat{A}$ and $\hat{B}$ are Hermitian then is the operator $[\hat{A}, \hat{B}]$ Hermitian or not? [2]

3. A triangle hat wave function is given by

$$\psi(x) = A\frac{x}{a};\;\; x \in [0, a]$$
$$= A\frac{(b-x)}{b-a};\; x \in [a, b]$$
$$= 0;\ \text{otherwise}$$

where A, a and b are constants.
(i) Sketch the wave function.
(ii) Determine the normalization constant A in terms of a and b. Also calculate $\langle x\rangle$. [2+5]

4. Show that if two operators commute, they have common eigenfunctions. [2]

5. The parity operator P operates on a function $f(x)$ in the following way: $Pf(x) = f(-x)$. Given that P and the Hamiltonian H commute and $\psi(x)$ is a solution of the time-independent Schrödinger equation, show that $\psi(-x)$ is a solution too with the same eigen-energy as $\psi(x)$. Find out the eigenvalues of the parity operator. [2+2]

CU – 2012

1. Show that $\exp(ikx)$ is an eigenfunction of the momentum operator in one dimension. Find out the corresponding eigenvalue. [2]

2. A particle can be in two different states given by orthonormal wave functions ψ_1 and ψ_2. If the probability of being in state ψ_1 is $1/3$, find out the normalized wavefunction for the particle. [2]

3. Calculate the normalization constant and the probability density for a wave function given by (at $t = 0$)

$$\psi(x) = A \exp\left(-\frac{\sigma^2 x^2}{2}\right) \exp(ikx)$$

4. Prove that $[A, BC] = B[A, C] + [A, B]C$

5. Prove the following commutator relations: [5]

(i) $[L_x, L_y] = i\hbar L_z$ (ii) $[L_x, L_z] = -i\hbar L_y$ (iii) $[L_x, L^2] = 0$

CU – 2011

1. If two operators $\hat{A}$ and $\hat{B}$ commute, show that they will have simultaneous eigenfunctions. [2]

2. Show that the eigenvalues of a Hermitian operator are real. [2]

3. If $\hat{H} = \frac{p^2}{2m} + V(x)$, show that $\left[x, \left[x, \hat{H}\right]\right] = -\frac{\hbar^2}{2m}$. [2]

4. A one-dimensional wave function is given by $\psi(x) = \sqrt{a}e^{-ax}$, find the probability of finding the particle between $x = 1/a$ and $x = 2/a$. [2]

5. For the wave function $\psi = A \exp i(ax - \omega t)$, find the probability current density. (A = const.) [2]

6. Find the constant B which makes $e^{-\alpha x^2}$ an eigenfunction of the operator $\left(\frac{d^2}{dx^2} - Bx^2\right)$. [2]

7. Evaluate $\left[\hat{L}_x^2 + \hat{L}_y^2, \hat{L}_z^2\right]$. [2]