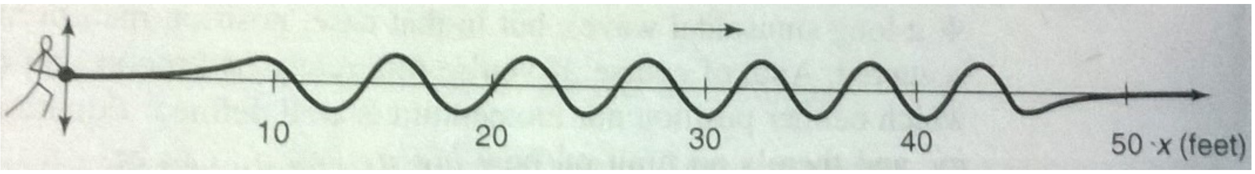
Uncertainty Principle

Hold one end of a very long rope and generate a wave by shaking it up and down rhythmically



Where is the wave precisely?

🤓 What a nutty question! The wave isn't precisely anywhere – it's spread out over 50 ft!

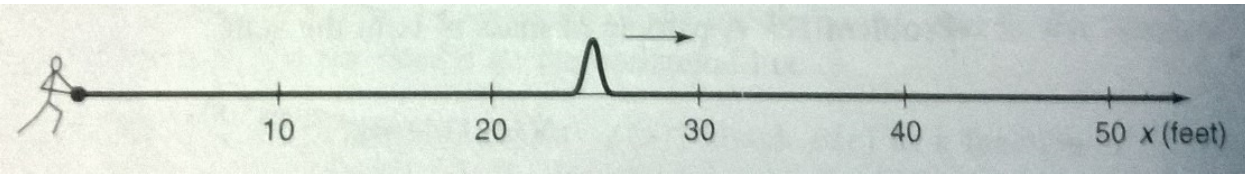
What is the wavelength?

😀 It's around 6 ft!

Introduction to Quantum Mechanics: D. J. Griffiths

Uncertainty Principle

Hold one end of the rope and give a sudden jerk!



Where is the wave precisely?

😀 It's somewhere around $x = 25$ ft

What is the wavelength?

🤓 How can you assign a wavelength to it? It isn't even vaguely periodic

Introduction to Quantum Mechanics: D. J. Griffiths

Uncertainty Principle

- ❑ To conclude, a wave with a (fairly) well-defined wavelength has ill-defined position; and a wave with (fairly) well-defined position has ill-defined wavelength
- ❑ The same speculation holds to any wave phenomenon, and hence to quantum mechanical wave function in particular
- ❑ For a quantum mechanical wave function, the wavelength is related to the momentum of the particle – de Broglie formula: $p = h/\lambda = \hbar k$
- ❑ A spread in wavelength thus corresponds to a spread in momentum, and our general observation now says that more precisely determined a particle's position is, the less precisely is its momentum; and vice-versa

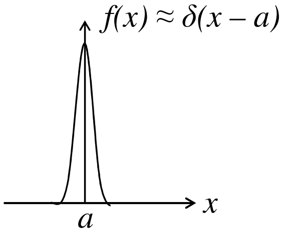
Uncertainty Principle

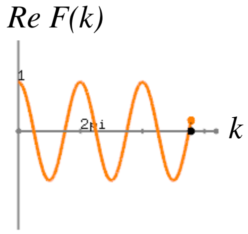
Quantitatively, $\Delta \boldsymbol{x}.\ \Delta \boldsymbol{p} \geq \hbar/2$

- ❑ The 'uncertainty' or the 'spread' refers to the fact that measurements on identically prepared systems do not yield identical results. In other words, repeated measurements on an identically prepared system do not yield same results
- ❑ If you construct a state such that repeated position measurements will be very close together (for example, consider ψ as Dirac-delta function – a localized 'spike') – i.e. Δx very small or zero. However, you have to pay the price if you want to measure the momentum in the same state – the measurements will be widely scattered i.e. Δp very large or infinity

Uncertainty Principle

- ❑ Alternatively, you could construct a state with reproducible momentum (by making $\psi(x)$ a long sinusoidal wave), but position measurement will be widely scattered

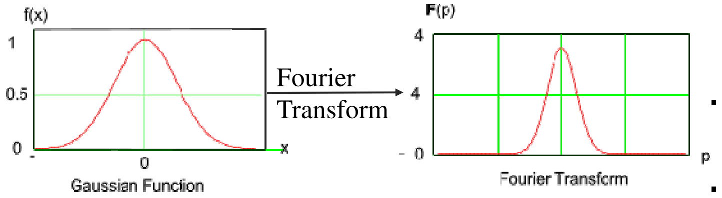




$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x-a) e^{ikx} dx = \frac{1}{\sqrt{2\pi}} e^{ika}$$

Uncertainty Principle

- ❑ Consider now a Gaussian wave function: $f(x) = \sqrt{\frac{\alpha}{\pi}} e^{-\alpha x^2} \xrightarrow[\text{Transform}]{\text{Fourier}} F(k) = \frac{1}{\sqrt{2\alpha}} e^{-\frac{k^2}{4\alpha}}$



- ❑ The uncertainty in measurement of position $\Delta x = \frac{1}{\sqrt{2\alpha}}$
- ❑ The uncertainty in measurement of momentum $\Delta k = (8\pi)^{\frac{1}{4}} \sqrt{\alpha}$

$$\boxed{\Delta x . \Delta k = \text{const.}}$$

Uncertainty Principle

Elementary Proof:

Consider there are n waves in a given length Δx: $\boxed{\Delta x = n\lambda}$ (λ is mean wavelength)

For another wavelength $\lambda + \Delta\lambda$ confined within Δx, $\Delta x = (n - \Delta n)(\lambda + \Delta\lambda)$

$$n\lambda + n.\,\Delta\lambda - \lambda.\,\Delta n - \Delta n.\,\Delta\lambda = n\lambda$$

Neglecting $\Delta n.\,\Delta\lambda$, $\boxed{\Delta\lambda/\lambda = \Delta n/n}$

Minimum of $\Delta n = 1$: $\boxed{\Delta\lambda/\lambda = 1/n}$

$p = h/\lambda$, $\Delta p = (h/\lambda^2)\,\Delta\lambda$

$\Delta p/p = \Delta\lambda/\lambda = 1/n$

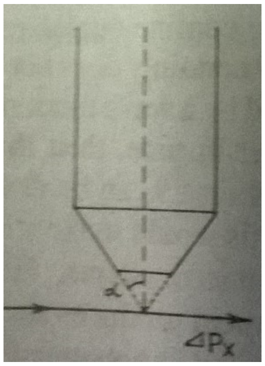
$\boxed{\Delta p = p/n}$

$$\boldsymbol{\Delta x.\,\Delta p = n\lambda.\,p/n = \lambda p = h}$$

Since $\Delta n \geq 1$, $\boldsymbol{\Delta x.\,\Delta p} \geq h$

Uncertainty Principle

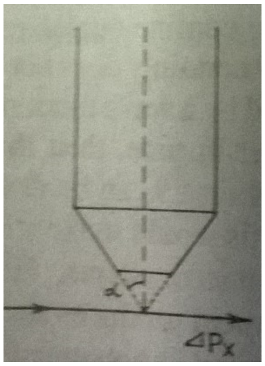
γ – ray microscope experiment: A hypothetical experiment



- An experiment to detect an electron – proposed by Niels Bohr
- To locate it as exactly as possible, Bohr assumed that a very short wavelength radiation like γ – ray may be used to illuminate the electron and this radiation scattered from the electron may then be observed by 'γ – ray microscope' (ordinary optical parts can't focus γ – rays)
- Resolving power of a microscope $\Delta x = \lambda/\sin\alpha$, $\sin\alpha$ is the measure of numerical aperture of the microscope
- Δx gives the uncertainty in the measurement of position of the electron

Uncertainty Principle

γ – ray microscope experiment: A hypothetical experiment



- γ – rays of wavelength λ possess momentum $p = h/\lambda$
- Scattered γ – ray of same momentum p has its x – component of momentum between 0 and $p \sin \alpha$
- The uncertainty in the definition of momentum of the γ – photon and hence that of electron is thus $\Delta p = p \sin \alpha$
- Hence $\Delta x \,.\, \Delta p = (\lambda/\sin \alpha).\,(p \sin \alpha) = \lambda/p = h$

These uncertainties are inherent in the physical world and have nothing to do with the skill of the observer

Introductory Quantum Mechanics: S. N. Ghoshal

Uncertainty Principle

Example of a base-ball:

- A pitcher throws a 0.1-kg baseball at 40 m/s
- So momentum is $0.1 \times 40 = 4$ kg m/s
- Suppose the momentum is measured to an accuracy of 1% , i.e. $\Delta p = 0.01p = 4 \times 10^{-2}$ kg-m/s
- The uncertainty in position is then

$$\Delta x \approx \frac{h}{\Delta p} \approx 10^{-32}\ \mathrm{m}$$

- No wonder one does not observe the effects of the uncertainty principle in everyday life!

Uncertainty Principle

Example of an electron :

- ☐ Same situation, but baseball replaced by an electron which has mass 9.11 × 10^{-31} kg traveling at 40 m/s
- ☐ So momentum is 3.6 × 10^{-29} kg m/s and its uncertainty is Δp = 3.6 × 10^{-31} kg m/s
- ☐ The uncertainty in position is then

$$\Delta x \approx \frac{h}{\Delta p} \approx 2 \times 10^{-3}\ \mathrm{m}$$

Uncertainty Principle

Applications: Impossibility of electron residing inside nucleus

If we assume that an electron resides inside nucleus, uncertainty in measurement of its position has its maximum value equal to nuclear diameter which is of the order of 10^{-14} m i.e. $\Delta x_{max} = 10^{-14}$ m

From Heisenberg's uncertainty principle, minimum uncertainty in measurement of linear momentum

$$\Delta p_{min} = \frac{\hbar}{2 \cdot \Delta x_{max}} = \frac{1.055 \times 10^{-34}}{2 \times 10^{-14}}\ \mathrm{kg\ m\ s^{-1}} \approx 5 \times 10^{-21}\ \mathrm{kg\ m\ s^{-1}}$$

Minimum value of linear momentum $p_{min} \approx \Delta p_{min} = 5 \times 10^{-21}$ kg m s^{-1}.

Hence the minimum kinetic energy of the electron if it resides inside the nucleus

$$E_{min} = \frac{p_{min}^2}{2m_e} \approx \frac{(5 \times 10^{-21})^2}{2 \times 9.1 \times 10^{-31}}\ \mathrm{J} = 1.37 \times 10^{-11}\ \mathrm{J} = 85.9\ \mathrm{MeV}$$

Experimental evidence, however, shows that $\beta -$ particles have energy around few MeV. Thus it can be concluded that an electron doesn't reside inside a nucleus.

Uncertainty Principle

Applications: Estimation of ground state energy of a particle in a box (1D)

For a particle confined in a one dimensional box of length l, the maximum uncertainty in the measurement of its position $\Delta x_{max} = l$.

From Heisenberg's uncertainty principle, the minimum uncertainty in the measurement of the linear moment of the particle is

$$\Delta p_{min} = \frac{\hbar}{2 \cdot \Delta x_{max}} = \frac{\hbar}{2l}$$

Thus the minimum linear momentum of the particle $p_{min} \approx \Delta p_{min} = \hbar/2l$.

The minimum energy corresponding to this momentum is

$$E_{min} = \frac{p_{min}^2}{2m} = \frac{1}{2m}\left(\frac{\hbar}{2l}\right)^2 = \frac{\hbar^2}{8ml^2}$$

Uncertainty Principle

Applications: Estimation of ground state energy of a simple harmonic oscillator

Consider a simple harmonic oscillator of mass m, amplitude a and angular frequency ω. The oscillator must be found between $-a$ and a and hence $\Delta x_{max} = 2a$.

$$\therefore \Delta p_{min} = \frac{\hbar}{2 \cdot \Delta x_{max}} = \frac{\hbar}{4a}$$

$$\therefore E = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 a^2 = \frac{\hbar^2}{32ma^2} + \frac{1}{2}m\omega^2 a^2$$

Form E to be minimum, $dE/da = 0$ i.e.

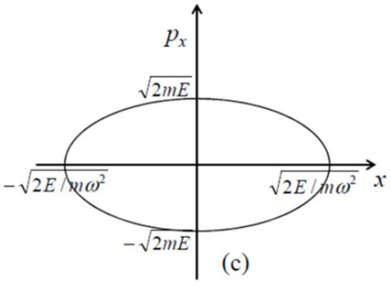
$$\frac{\hbar^2}{32m}\left(-\frac{2}{a^3}\right) + m\omega^2 a = 0 \Rightarrow a^2 = \frac{\hbar}{4m\omega}$$

$$\therefore E_{min} = E\left(\text{at } a = \sqrt{\frac{\hbar}{4m\omega}}\right) = \frac{\hbar^2}{32m}\left(\frac{4m\omega}{\hbar}\right) + \frac{1}{2}m\omega^2\left(\frac{\hbar}{4m\omega}\right) = \frac{\hbar\omega}{4}$$

Uncertainty Principle

Applications: Impossibility of a particle following a trajectory

- Trajectory of a particle requires complete information on position and momentum simultaneously which is forbidden by the uncertainty principle
- Consider, for example, the trajectory of a classical simple harmonic oscillator. Every point on the trajectory corresponds to definite values of position and momentum at a given instant i.e. $\Delta x = \Delta p = 0$ which is contradictory to what follows from the uncertainty principle



Uncertainty Principle

Problem: Uncertainty principle forces us to reject Bohr's atomic model

As the electron moves in a circular orbit (radius a), position-momentum uncertainty principle will apply to the tangential directions i.e. $\Delta p.\Delta s \geq \hbar/2$

The angular momentum $L = mva = pa$ and the uncertainty $\Delta L = \Delta p.a$ or $\Delta p = \Delta L/a$.

Again, the angular displacement θ is related to the arc length s as $s = a\theta$. Hence, $\Delta s = a\Delta\theta$.

$$\therefore (\Delta L/a).(a\Delta\theta) \geq \frac{\hbar}{2} \quad \Rightarrow \Delta L.\Delta\theta \geq \frac{\hbar}{2}$$

According to Bohr's model, the angular momentum of the electron is precisely defined i.e. $\Delta L = 0$. Therefore, the uncertainty principle predicts $\Delta\theta = \infty$ i.e. the position of the electron is indeterminate.

Uncertainty Principle

Problem: $\Delta x.\Delta p \geq \hbar/2$ leads to the energy-time uncertainty relation $\Delta E.\Delta t \geq \hbar/2$

The kinetic energy E of the particle of mass m is related to the linear momentum p as

$$E = \frac{p^2}{2m}$$

The uncertainty in energy

$$\Delta E = \frac{p\Delta p}{m} = \frac{mv\Delta p}{m} = v\Delta p$$

If uncertainty in measurement of position is Δx, it is related to uncertainty in time Δt as $\Delta x = v\Delta t$

Thus from the position-momentum uncertainty relation $\Delta x.\Delta p \geq \hbar/2$ it follows that

$$v\Delta t.\frac{\Delta E}{v} \geq \frac{\hbar}{2}$$

$$\text{or,} \qquad \Delta E.\Delta t \geq \frac{\hbar}{2}$$

Uncertainty Principle

Summary:

- Heisenberg's uncertainty principle in quantum mechanics asserts a fundamental limit to the precision with which certain pairs of physical properties of a system (complementary variables) can be known
- These uncertainties are inherent in physical world and have nothing to do with skill of the observer
- Uncertainty relation between two complementary variables position & momentum: $\Delta x \cdot \Delta p \geq \hbar/2$
- Uncertainty relation between angular momentum & angular position: $\Delta L.\Delta\phi \geq \hbar/2$
- Uncertainty relation between a pair of complementary variables energy & time is $\Delta E.\Delta t \geq \hbar/2$
- According to the statistical definition, the uncertainty of a physical property is the standard deviation. For example, the uncertainties in position and momentum are respectively defined as

$$\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2}$$