2024

MATHEMATICS — HONOURS

Paper : DSCC-1

(Calculus, Geometry and Vector Analysis)

Full Marks : 75

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

[Calculus]

(Marks : 20)

1. Answer ***any two*** questions : 2×2

(a) Find a and b in order that

$$\lim_{x \to 0} \frac{a\sin 2x - b\sin x}{x^3} = 1$$

(b) If $I_n = \int_0^1 \left(1 - x^2\right)^n dx$, then prove that $(2n + 1)I_n = 2n\, I_{n-1}$.

(c) Find the length of the arc of the curve $y = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$ between the points $x = 0$, $x = 3$.

2. Answer ***any four*** questions :

(a) Prove that

$$\lim_{x \to \infty} \left(\frac{a_1^{1/x} + a_2^{1/x} + ... + a_n^{1/x}}{n}\right)^{nx} = a_1 a_2 ..... a_n$$ 4

(b) If $y^{1/m} + y^{-1/m} = 2x$, prove that

(i) $(x^2 - 1)\, y_2 + xy_1 - m^2 y = 0$

(ii) $(x^2 - 1)\, y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 - m^2)\, y_n = 0$ 2+2

(c) If $I_{m,n} = \int_0^{\frac{\pi}{2}} \cos^m x \cos nx\, dx$, then show that $\left(m^2 - n^2\right) I_{m,n} - m(m-1)\, I_{m-2,n} = 0$. 4

(d) Show that the length of the parabola $y^2 = 4ax$ cut off by its latus rectum is $2a\left[\sqrt{2} + \log\left(1 + \sqrt{2}\right)\right]$. 4

(e) Two curves $y = 4x^2$ and $y^2 = 2x$ passing through the origin form a loop at another point P. Prove that the line OP divides the loop into two parts of equal area. 4

(f) Prove that the volume of the solid generated by revolving the region bounded by the curve $y = \log x$, $x = 2$ and the x-axis about the x-axis is $2\pi(1 - \log 2)^2$. 4

Group - B

[Geometry]

(Marks : 35)

3. Answer ***any two*** questions : 2½×2

(a) If PSP' be a focal chord of a conic, then show that $\frac{1}{SP} + \frac{1}{SP'} = \frac{2}{l}$, where l is the semi-latus rectum.

(b) Find the equations to the tangents to the conic $x^2 + 4xy + 3y^2 - 5x - 6y + 3 = 0$, which are parallel to $x + 4y = 0$.

(c) Find the equation of the circle on the sphere $x^2 + y^2 + z^2 = 49$ whose centre is at $(2, -1, 3)$.

4. Answer ***any five*** questions :

(a) Reduce the equation $7x^2 - 6xy - y^2 + 4x - 4y - 2 = 0$ to its canonical form and find the nature of the conic. 6

(b) If the normal is drawn at one extremity of latus rectum PSP′ of the conic $\frac{l}{r} = 1 + e\cos\theta$, where S is the pole, show that the distance from the focus S to the other point in which the normal meets the curve is $\frac{l\left(1 + 3e^2 + e^4\right)}{1 + e^2 - e^4}$. 6

(c) The tangents at the extremities of a normal chord of the parabola $y^2 = 4ax$ meet in a point T. Show that locus of T is the curve $(x + 2y)y^2 = 4a^3$. 6

(d) (i) Find the equation of the sphere through the circle $x^2 + y^2 + z^2 - 4x - 6y + 2z - 16 = 0$, $3x + y + 3z - 4 = 0$ such that the point $(1, 0, -3)$ lies on the sphere.

(ii) A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere passing through the origin and the points A, B, C is $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$. 3+3

(e) Find the equation of the cylinder whose guiding curve is the ellipse $4x^2 + y^2 = 1$, $z = 0$ and generators are parallel to the straight line $x/2 = y/-1 = z/3$. 6

(f) Find the equations of the generators of the conicoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, through a point of the principal elliptic section by the plane $z = 0$. 6

(g) Find the equation of the hyperboloid through the three lines $y - z = 1, x = 0$; $z - x = 1, y = 0$; and $x - y = 1, z = 0$. Also obtain the equations of the two system of generators. 6

(h) The section of a cone whose guiding curve is the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$ by the plane $x = 0$ is a rectangular hyperbola. Show that the locus of the vertex is $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$. 6

Group - C

[Vector Analysis]

(Marks : 20)

5. Answer ***any two*** questions : 2×2

(a) Show that if $\vec{a}, \vec{b}, \vec{c}$ be three vectors, then $\left[\vec{a} + \vec{b} \;\; \vec{b} + \vec{c} \;\; \vec{c} + \vec{a}\right] = 2\left[\vec{a}\;\vec{b}\;\vec{c}\right]$.

(b) Solve for $\vec{r}$ from the vector equation $p\vec{r} + \left(\vec{r} \cdot \vec{b}\right)\vec{a} = \vec{c}$, where p is known non-zero scalar, $\vec{a}, \vec{b}, \vec{c}$ are known vectors and $p + \vec{a} \cdot \vec{b} \neq 0$.

(c) A rigid body is spinning with angular velocity of 5 radians per second about an axis with direction $3\hat{j} - \hat{k}$, passing through the point A (1, 3, –1). Find the velocity of the particle at the point P (4, –2, 1).

6. Answer ***any four*** questions :

(a) Prove that $\left(\vec{a} \times \vec{b}\right) \times \left(\vec{c} \times \vec{d}\right) = \left[\vec{a}\;\vec{c}\;\vec{d}\right]\vec{b} - \left[\vec{b}\;\vec{c}\;\vec{d}\right]\vec{a} = \left[\vec{a}\;\vec{b}\;\vec{d}\right]\vec{c} - \left[\vec{a}\;\vec{b}\;\vec{c}\right]\vec{d}$.

Hence express $\vec{d}$ in terms of the non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$. 3+1

(b) Find the point of intersection of the straight line joining the points with position vector $3\hat{i} + 6\hat{j} - 5\hat{k}$ and $\hat{i} + 2\hat{j} + 3\hat{k}$ and the plane passing through the points (1, –2, 4), (3, 0, 2) and (3, 1, 4). 4

(c) Using vector method find the distance of the point (2, 3, 4) from the plane $3x - 6y + 2z + 11 = 0$ measured along the vector $3\hat{i} + \hat{j} + 2\hat{k}$. 4

(d) (i) Prove that for any proper vector $\vec{\alpha}$,

$$\hat{i} \times (\vec{\alpha} \times \hat{i}) + \hat{j} \times (\vec{\alpha} \times \hat{j}) + \hat{k} \times (\vec{\alpha} \times \hat{k}) = 2\vec{\alpha}.$$

(ii) Find the moment about the point A(1, 2, 3) of a force of magnitude 5 units acting through the point (3, 4, 5) in the direction of the vector $2\hat{i} + 3\hat{j} + 4\hat{k}$. 2+2

(e) The acceleration of a moving particle at any time t is given by

$$\frac{d^2\vec{r}}{dt^2} = (12\cos 2t)\hat{i} - (8\sin 2t)\hat{j} + (16t)\hat{k}.$$

Find the velocity $(\vec{v})$ and displacement $(\vec{r})$ at any time t, if it is given that at $t = 0, \vec{v} = \vec{0}, \vec{r} = \vec{0}$. 2+2

(f) Find the value of $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$ for the curve $\vec{r}(t) = (3t, 2t^2, 3t^3)$. 4
