DIFFERENTIAL OPERATORS
(Vector Calculus)

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Def.^{n} :- Scalar field :

At each point P of a certain region R, we may associate (by any law) a scalar denoted by $f(P)$, then we say that f is a scalar point function for the region R. The point P in the region R together with the function value $f(P)$ will form a **scalar field** over R.

 # Examples : The temperature T within a body B is a scalar field, namely, the temperature field. The another examples are mass distribution in a body, electric potential of a system of charges, etc.

Def.^{n} :- Vector field :

Let a vector $\overrightarrow{F}(P)$ be assigned to each point P of a set of points in space either lying on a curve, a surface or a three dimensional region, then $\overrightarrow{F}(P)$ is called a vector function and we can say a **vector field** is defined at those set of points.

 # Examples : Examples of vector fields are velocity field of a moving particle, force field defined by forces acting on a body, gravitational field defined by a system of particles under the action of gravity.

Gradient of a scalar point function :

Let a scalar field be defined by the scalar point function $\phi(x, y, z)$ of the co-ordinates $x,\ y,\ z$ which is also defined and differentiable at each point (x, y, z) in some region of space. The vector function $\widehat{i}\ \frac{\partial \phi}{\partial x} + \widehat{j}\ \frac{\partial \phi}{\partial y} + \widehat{k}\ \frac{\partial \phi}{\partial z}$ is called the **gradient of the scalar point function** ϕ, where $\widehat{i}$, $\widehat{j}$ and $\widehat{k}$ are three mutually perpendicular non-coplanar unit vectors.
This gradient is frequently written in operational notation as

$$\text{grad } \phi = \left(\widehat{i}\ \frac{\partial}{\partial x} + \widehat{j}\ \frac{\partial}{\partial y} + \widehat{k}\ \frac{\partial}{\partial z} \right) \phi \ .$$

Using the differential operator

$$\overrightarrow{\nabla} = \widehat{i}\ \frac{\partial}{\partial x} + \widehat{j}\ \frac{\partial}{\partial y} + \widehat{k}\ \frac{\partial}{\partial z} \ , \ \text{we may write} \ \ \text{grad } \phi = \overrightarrow{\nabla} \phi \ .$$

Geometrical interpretation of $\overrightarrow{\nabla}\phi$:

We consider a surface represented by the equation $\phi(x, y, z) = c$ (constant),
$\Rightarrow \quad d\phi = 0$, $\Rightarrow$ $\frac{\partial \phi}{\partial x}\ dx + \frac{\partial \phi}{\partial y}\ dy + \frac{\partial \phi}{\partial z}\ dz = 0$,

$\Rightarrow \quad \left(\widehat{i}\,\frac{\partial\phi}{\partial x}+\widehat{j}\,\frac{\partial\phi}{\partial y}+\widehat{k}\,\frac{\partial\phi}{\partial z}\right)\cdot(\widehat{i}\,dx+\widehat{j}\,dy+\widehat{k}\,dz)=0$,
$\Rightarrow \; \overrightarrow{\nabla}\phi\cdot d\overrightarrow{r}=0$,
where $\overrightarrow{r}=x\;\widehat{i}+y\;\widehat{j}+z\;\widehat{k}$ is the position vector of any point P on the surface $\phi(x,y,z)=c$. Now $d\overrightarrow{r}$ is the arbitrary vector along the tangent plane to the surface $\phi(x,y,z)=c$ at the point (x,y,z).
Since $\overrightarrow{\nabla}\phi$ is perpendicular to arbitrary line on the tangent plane to the surface $\phi(x,y,z)=c$ at the point $P(x,y,z)$, so $\overrightarrow{\nabla}\phi$ is perpendicular to the tangent plane. Hence $\overrightarrow{\nabla}\phi$ is normal to the point at (x,y,z).

Def.^{n} :- Directional derivative :

Let $\phi(x,y,z)$ be a scalar point function and $\widehat{a}$ be a given unit vector. Then the directional derivative of ϕ along the unit vector $\widehat{a}$ at the point (x_0,y_0,z_0) is defined by $\overrightarrow{\nabla}\phi(x_0,y_0,z_0)\cdot\widehat{a}$.
The maximum value of the directional derivative of ϕ at the point (x_0,y_0,z_0) is given by $|\overrightarrow{\nabla}\phi(x_0,y_0,z_0)|$.
The direction of ϕ at (x_0,y_0,z_0) attained in maximum value is given by $\frac{\overrightarrow{\nabla}\phi(x_0,y_0,z_0)}{|\overrightarrow{\nabla}\phi(x_0,y_0,z_0)|}$.

Def.^{n} :- Divergence of a vector field :

Let $\overrightarrow{F}(x,y,z)$ be a differentiable vector function of the Cartesian co-ordinates (x,y,z) in space and $\overrightarrow{F}=F_1\;\widehat{i}+F_2\;\widehat{j}+F_3\;\widehat{k}$. Then divergence of $\overrightarrow{F}$, denoted by div$\overrightarrow{F}$ and defined by

$$\text{div}\overrightarrow{F}=\overrightarrow{\nabla}\cdot\overrightarrow{F}=\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}+\frac{\partial F_3}{\partial z}\;,\;\text{which is the scalar function of }\;x,\;y,\;z\;.$$

Note-1 : $\overrightarrow{\nabla}\cdot\overrightarrow{F}\neq\overrightarrow{F}\cdot\overrightarrow{\nabla}$.

Def.^{n} :- Solenoidal vector :

A vector field $\overrightarrow{F}$ is said to be **solenoidal vector**, if div $\overrightarrow{F}=\overrightarrow{\nabla}\cdot\overrightarrow{F}=0$ everywhere in the space considered.

Example : If we take $\overrightarrow{F}=y^2z^2\widehat{i}+z^2x^2\widehat{j}+x^2y^2\widehat{k}$, then

$$\text{div }\overrightarrow{F}=\overrightarrow{\nabla}\cdot\overrightarrow{F}=\frac{\partial}{\partial x}(y^2z^2)+\frac{\partial}{\partial y}(z^2x^2)+\frac{\partial}{\partial z}(x^2y^2)=0\;.$$

So the vector field $\overrightarrow{F}$ is a **solenoidal vector**.

Def.^{n} :- Curl of a vector field :

The **curl** of a vector function $\vec{F} = F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k}$ is a vector function denoted by curl $\vec{F}$ (or rot $\vec{F}$) and is defined by

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \left(\hat{i}\,\frac{\partial\phi}{\partial x} + \hat{j}\,\frac{\partial\phi}{\partial y} + \hat{k}\,\frac{\partial\phi}{\partial z}\right) \times (F_1\,\hat{i} + F_2\,\hat{j} + F_3\,\hat{k})$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\hat{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\hat{k}\,.$$

Def.^{n} :- Irrotational vector :

A vector field $\vec{F}$ is said to be **irrotational vector**, if curl $\vec{F} = \vec{\nabla} \times \vec{F} = \vec{0}$ everywhere in the space considered.

Example : If we take $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$, then

$$\text{curl } \vec{F} = \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$= \left(\frac{\partial z^3}{\partial y} - \frac{\partial y^3}{\partial z}\right)\hat{i} + \left(\frac{\partial x^3}{\partial z} - \frac{\partial z^3}{\partial x}\right)\hat{j} + \left(\frac{\partial y^3}{\partial x} - \frac{\partial x^3}{\partial y}\right)\hat{k} = \vec{0}\,.$$

So the vector field $\vec{F}$ is an **irrotational vector**.

Def.^{n} :- Laplacian operator :

The scalar differential operator $\vec{\nabla}^2 = \vec{\nabla} \cdot \vec{\nabla}$, where $\vec{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z}$ is called the **Laplacian operator**.
If ϕ is a continuously differentiable scalar point function of $x,\ y,\ z$, then

$$\vec{\nabla}^2\phi = \left(\vec{\nabla} \cdot \vec{\nabla}\right)\phi = \vec{\nabla} \cdot \left(\vec{\nabla}\phi\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\,.$$

Def.^{n} :- Harmonic function & Laplace's equation :

If ψ be continuously differentiable scalar point function of $x,\ y,\ z$ and is satisfies the equation $\nabla^2\psi = 0$, then ψ is called a **harmonic function** and the equation $\nabla^2\psi = 0$ is called **Laplace's equation**.

 # Note-2 : If $\vec{F}$ be a vector point function, then $\vec{\nabla}^2\vec{F}$ will mean $\vec{\nabla}\left(\vec{\nabla}\cdot\vec{F}\right)$ or grad $\left(\text{div}\,\vec{F}\right)$.

 # Question :- Prove that for vector functions $\vec{A}$, $\vec{B}$ and a scalar function ϕ,

(i) $\text{div}(\phi\vec{A}) = \phi\ \text{div}\vec{A} + \vec{A}\cdot\text{grad}\ \phi$ i.e., $\vec{\nabla}\cdot(\phi\vec{A}) = \phi\ \vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\phi$.

(ii) $\text{curl}(\phi\vec{A}) = \phi\ \text{curl}\vec{A} + \text{grad}\ \phi\times\vec{A}$ i.e., $\vec{\nabla}\times(\phi\vec{A}) = \phi\ \vec{\nabla}\times\vec{A} + \vec{\nabla}\phi\times\vec{A}$.

(iii) $\text{grad}(\vec{A}\cdot\vec{B}) = \vec{A}\times\text{curl}\vec{B} + \vec{B}\times\text{curl}\vec{A} + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$
i.e., $\vec{\nabla}(\vec{A}\cdot\vec{B}) = \vec{A}\times(\vec{\nabla}\times\vec{B}) + \vec{B}\times(\vec{\nabla}\times\vec{A}) + (\vec{A}\cdot\vec{\nabla})\vec{B} + (\vec{B}\cdot\vec{\nabla})\vec{A}$.

(iv) $\text{div}(\vec{A}\times\vec{B}) = \vec{B}\cdot\text{curl}\vec{A} - \vec{A}\cdot\text{curl}\vec{B}$ i.e., $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\vec{\nabla}\times\vec{A}) - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$.

(v) $\text{curl}(\vec{A}\times\vec{B}) = \vec{\nabla}\times(\vec{A}\times\vec{B}) = \vec{A}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{A}) + (\vec{B}\cdot\vec{\nabla})\vec{A} - (\vec{A}\cdot\vec{\nabla})\vec{B}$.

(vi) $\text{curl grad}\ \phi = \vec{0}$ i.e., $\vec{\nabla}\times(\vec{\nabla}\phi) = \vec{0}$.

(vii) $\text{div curl}\ \vec{A} = 0$ i.e., $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$.

$(viii)$ $\text{curl curl}\vec{A} = \text{grad div}\vec{A} - \vec{\nabla}^2\vec{A}$ i.e., $\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}^2\vec{A}$.

 # Note-3 : $\vec{\nabla} = \hat{i}\,\frac{\partial}{\partial x} + \hat{j}\,\frac{\partial}{\partial y} + \hat{k}\,\frac{\partial}{\partial z} = \sum\hat{i}\,\frac{\partial}{\partial x}$.

 # Answer :- (i) $\vec{\nabla}\cdot\left(\phi\vec{A}\right) = \phi\ \vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}\phi$.

$$\vec{\nabla}\cdot\left(\phi\vec{A}\right) = \left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\cdot\left(\phi\vec{A}\right) = \sum\hat{i}\,\frac{\partial}{\partial x}\cdot\left(\phi\vec{A}\right) = \sum\hat{i}\cdot\frac{\partial}{\partial x}\left(\phi\vec{A}\right)$$

$$= \sum\hat{i}\cdot\left\{\frac{\partial\phi}{\partial x}\,\vec{A} + \phi\,\frac{\partial\vec{A}}{\partial x}\right\} = \sum\hat{i}\cdot\frac{\partial\phi}{\partial x}\,\vec{A} + \sum\hat{i}\cdot\phi\,\frac{\partial\vec{A}}{\partial x}$$

$$= \sum\hat{i}\,\frac{\partial\phi}{\partial x}\cdot\vec{A} + \phi\sum\hat{i}\cdot\frac{\partial\vec{A}}{\partial x} = \left(\sum\hat{i}\,\frac{\partial\phi}{\partial x}\right)\cdot\vec{A} + \phi\left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\cdot\vec{A}$$

$$= \left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\phi\cdot\vec{A} + \phi\ \vec{\nabla}\cdot\vec{A} = \vec{\nabla}\ \phi\cdot\vec{A} + \phi\ \vec{\nabla}\cdot\vec{A}\ .$$

Hence $\vec{\nabla}\cdot\left(\phi\vec{A}\right) = \vec{\nabla}\ \phi\cdot\vec{A} + \phi\ \vec{\nabla}\cdot\vec{A}$. (**Proved**)

 # Answer :- (ii) $\vec{\nabla}\times\left(\phi\vec{A}\right) = \phi\ \vec{\nabla}\times\vec{A} + \vec{\nabla}\phi\times\vec{A}$.

$$\vec{\nabla}\times\left(\phi\vec{A}\right) = \left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\times\left(\phi\vec{A}\right) = \sum\hat{i}\,\frac{\partial}{\partial x}\times\left(\phi\vec{A}\right) = \sum\hat{i}\times\frac{\partial}{\partial x}\left(\phi\vec{A}\right)$$

$$= \sum \hat{i} \times \left\{ \frac{\partial \phi}{\partial x} \vec{A} + \phi \frac{\partial \vec{A}}{\partial x} \right\} = \sum \hat{i} \times \frac{\partial \phi}{\partial x} \vec{A} + \sum \hat{i} \times \phi \frac{\partial \vec{A}}{\partial x}$$

$$= \sum \hat{i} \frac{\partial \phi}{\partial x} \times \vec{A} + \phi \sum \hat{i} \times \frac{\partial \vec{A}}{\partial x} = \left(\sum \hat{i} \frac{\partial \phi}{\partial x} \right) \times \vec{A} + \phi \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \times \vec{A}$$

$$= \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \phi \times \vec{A} + \phi \vec{\nabla} \times \vec{A} = \vec{\nabla} \phi \times \vec{A} + \phi \vec{\nabla} \times \vec{A} .$$

Hence $\vec{\nabla} \times \left(\phi \vec{A} \right) = \phi \vec{\nabla} \times \vec{A} + \vec{\nabla} \phi \times \vec{A}$. **(Proved)**

 # Answer :- *(iii)*
$\vec{\nabla}(\vec{A} \cdot \vec{B}) = \vec{A} \times (\vec{\nabla} \times \vec{B}) + \vec{B} \times (\vec{\nabla} \times \vec{A}) + (\vec{A} \cdot \vec{\nabla})\vec{B} + (\vec{B} \cdot \vec{\nabla})\vec{A}.$

$$\vec{\nabla} \left(\vec{A} \cdot \vec{B} \right) = \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \left(\vec{A} \cdot \vec{B} \right) = \sum \hat{i} \frac{\partial}{\partial x} \left(\vec{A} \cdot \vec{B} \right)$$

$$= \sum \hat{i} \left\{ \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} + \vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right\} = \sum \hat{i} \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} + \sum \hat{i} \frac{\partial \vec{B}}{\partial x} \cdot \vec{A} . \tag{1}$$

Now $\vec{A} \times \left(\vec{\nabla} \times \vec{B} \right) = \vec{A} \times \left\{ \left(\sum \hat{i} \frac{\partial}{\partial x} \right) \times \vec{B} \right\} = \vec{A} \times \left(\sum \hat{i} \times \frac{\partial \vec{B}}{\partial x} \right)$

$$= \sum \left\{ \vec{A} \times \left(\hat{i} \times \frac{\partial \vec{B}}{\partial x} \right) \right\} = \sum \left\{ \left(\vec{A} \cdot \frac{\partial \vec{B}}{\partial x} \right) \hat{i} - (\vec{A} \cdot \hat{i}) \frac{\partial \vec{B}}{\partial x} \right\}$$

$$= \sum \hat{i} \frac{\partial \vec{B}}{\partial x} \cdot \vec{A} - \left(\vec{A} \cdot \sum \hat{i} \frac{\partial}{\partial x} \right) \vec{B} = \sum \hat{i} \frac{\partial \vec{B}}{\partial x} \cdot \vec{A} - \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} ,$$

$$\Rightarrow \sum \hat{i} \frac{\partial \vec{B}}{\partial x} \cdot \vec{A} = \vec{A} \times \left(\vec{\nabla} \times \vec{B} \right) + \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} . \tag{2}$$

Similarly $\sum \hat{i} \frac{\partial \vec{A}}{\partial x} \cdot \vec{B} = \vec{B} \times \left(\vec{\nabla} \times \vec{A} \right) + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{A} .$ (3)

From (1), (2) and (3), we have

$\vec{\nabla} \left(\vec{A} \cdot \vec{B} \right) = \vec{A} \times \left(\vec{\nabla} \times \vec{B} \right) + \vec{B} \times \left(\vec{\nabla} \times \vec{A} \right) + \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} + \left(\vec{B} \cdot \vec{\nabla} \right) \vec{A}$. **(Proved)**

 # Answer :- *(iv)* $\vec{\nabla} \cdot \left(\vec{A} \times \vec{B} \right) = \vec{B} \cdot \left(\vec{\nabla} \times \vec{A} \right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B} \right).$

$$\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\cdot\left(\vec{A}\times\vec{B}\right)=\sum\hat{i}\cdot\,\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)$$

$$=\sum\hat{i}\cdot\left\{\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right\}=\sum\hat{i}\cdot\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\sum\hat{i}\cdot\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$=\left(\sum\hat{i}\times\frac{\partial\vec{A}}{\partial x}\right)\cdot\vec{B}-\sum\hat{i}\cdot\left(\frac{\partial\vec{B}}{\partial x}\times\vec{A}\right)=\left(\sum\hat{i}\,\frac{\partial}{\partial x}\times\vec{A}\right)\cdot\vec{B}-\left(\sum\hat{i}\times\frac{\partial\vec{B}}{\partial x}\right)\cdot\vec{A}$$

$$=\left(\vec{\nabla}\times\vec{A}\right)\cdot\vec{B}-\left(\sum\hat{i}\,\frac{\partial}{\partial x}\times\vec{B}\right)\cdot\vec{A}=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)\ .$$

Hence $\vec{\nabla}\cdot\left(\vec{A}\times\vec{B}\right)=\vec{B}\cdot\left(\vec{\nabla}\times\vec{A}\right)-\vec{A}\cdot\left(\vec{\nabla}\times\vec{B}\right)$. **(Proved)**

 # Answer :- (v)
$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}.$$

$$\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\times\left(\vec{A}\times\vec{B}\right)=\sum\hat{i}\times\,\frac{\partial}{\partial x}\left(\vec{A}\times\vec{B}\right)$$

$$=\sum\hat{i}\times\left\{\frac{\partial\vec{A}}{\partial x}\times\vec{B}+\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right\}=\sum\hat{i}\times\left(\frac{\partial\vec{A}}{\partial x}\times\vec{B}\right)+\sum\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)$$

$$=\sum\hat{i}\times\left(\vec{A}\times\frac{\partial\vec{B}}{\partial x}\right)-\sum\hat{i}\times\left(\vec{B}\times\frac{\partial\vec{A}}{\partial x}\right)$$

$$=\sum\left\{\left(\hat{i}\cdot\frac{\partial\vec{B}}{\partial x}\right)\vec{A}-\left(\hat{i}\cdot\vec{A}\right)\frac{\partial\vec{B}}{\partial x}\right\}-\sum\left\{\left(\hat{i}\cdot\frac{\partial\vec{A}}{\partial x}\right)\vec{B}-\left(\hat{i}\cdot\vec{B}\right)\frac{\partial\vec{A}}{\partial x}\right\}$$

$$=\left(\sum\hat{i}\,\frac{\partial}{\partial x}\cdot\vec{B}\right)\vec{A}-\left(\vec{A}\cdot\sum\hat{i}\,\frac{\partial}{\partial x}\right)\vec{B}-\left(\sum\hat{i}\,\frac{\partial}{\partial x}\cdot\vec{A}\right)\vec{B}+\left(\vec{B}\cdot\sum\hat{i}\,\frac{\partial}{\partial x}\right)\vec{A}$$

$$=\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}\ .$$

Hence $\vec{\nabla}\times\left(\vec{A}\times\vec{B}\right)=\vec{A}\left(\vec{\nabla}\cdot\vec{B}\right)+\left(\vec{B}\cdot\vec{\nabla}\right)\vec{A}-\vec{B}\left(\vec{\nabla}\cdot\vec{A}\right)-\left(\vec{A}\cdot\vec{\nabla}\right)\vec{B}\ .$

 # Answer :- (vi) $\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\vec{0}$.

$$\vec{\nabla}\times\left(\vec{\nabla}\phi\right)=\left(\sum\hat{i}\,\frac{\partial}{\partial x}\right)\times\left(\vec{\nabla}\phi\right)=\sum\hat{i}\times\frac{\partial}{\partial x}\left\{\hat{i}\,\frac{\partial\phi}{\partial x}+\hat{j}\,\frac{\partial\phi}{\partial y}+\hat{k}\,\frac{\partial\phi}{\partial z}\right\}$$

$$= \sum \widehat{i} \times \left\{ \widehat{i}\, \frac{\partial^2 \phi}{\partial x^2} + \widehat{j}\, \frac{\partial^2 \phi}{\partial x \partial y} + \widehat{k}\, \frac{\partial^2 \phi}{\partial x \partial z} \right\} = \sum \left\{ \widehat{k}\, \frac{\partial^2 \phi}{\partial x \partial y} - \widehat{j}\, \frac{\partial^2 \phi}{\partial x \partial z} \right\} = \overrightarrow{0}\ ,$$

using $\widehat{i} \times \widehat{i} = \overrightarrow{0}$, $\widehat{i} \times \widehat{j} = \widehat{k}$, etc.; assuming continuous differentiability of ϕ will imply $\phi_{xy} = \phi_{yx}$, $\phi_{xz} = \phi_{zx}$, $\phi_{yz} = \phi_{zy}$; terms like $\widehat{k}\, \frac{\partial^2 \phi}{\partial x \partial y}$ and $-\widehat{k}\, \frac{\partial^2 \phi}{\partial y \partial x}$ will cancel.

 # Answer :- (vii) $\overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right) = 0.$

$$\overrightarrow{\nabla} \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right) = \left(\sum \widehat{i}\, \frac{\partial}{\partial x}\right) \cdot \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right) = \sum \widehat{i} \cdot \frac{\partial}{\partial x} \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right)$$

$$= \sum \widehat{i} \cdot \frac{\partial}{\partial x} \left\{ \widehat{i} \times \frac{\partial \overrightarrow{A}}{\partial x} + \widehat{j} \times \frac{\partial \overrightarrow{A}}{\partial y} + \widehat{k} \times \frac{\partial \overrightarrow{A}}{\partial z} \right\}$$

$$= \sum \left\{ \widehat{i} \times \widehat{i} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x^2} + \widehat{i} \times \widehat{j} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x \partial y} + \widehat{i} \times \widehat{k} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x \partial z} \right\}$$

$$= \sum \left\{ \widehat{k} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x \partial y} - \widehat{j} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x \partial z} \right\} = 0\ ,$$

using $\widehat{i} \times \widehat{i} = \overrightarrow{0}$, $\widehat{i} \times \widehat{j} = \widehat{k}$, etc.; assuming continuous differentiability of $\overrightarrow{A}$ will imply $\frac{\partial^2 \overrightarrow{A}}{\partial x \partial y} = \frac{\partial^2 \overrightarrow{A}}{\partial y \partial x}$, $\frac{\partial^2 \overrightarrow{A}}{\partial x \partial z} = \frac{\partial^2 \overrightarrow{A}}{\partial z \partial x}$, $\frac{\partial^2 \overrightarrow{A}}{\partial y \partial z} = \frac{\partial^2 \overrightarrow{A}}{\partial z \partial y}$; terms like $\widehat{k} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial x \partial y}$ and $-\widehat{k} \cdot \frac{\partial^2 \overrightarrow{A}}{\partial y \partial x}$ will cancel.

 # Answer :- $(viii)$ $\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right) = \overrightarrow{\nabla} \left(\overrightarrow{\nabla} \cdot \overrightarrow{A}\right) - \overrightarrow{\nabla}^2 \overrightarrow{A}.$

Let $\overrightarrow{A} = A_1\, \widehat{i} + A_2\, \widehat{j} + A_3\, \widehat{k}$, where $A_1,\ A_2,\ A_3$ are real-valued differentiable functions of real variables $x,\ y,\ z$.

Then $$\overrightarrow{\nabla} \times \overrightarrow{A} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix} = L_1\, \widehat{i} + L_2\, \widehat{j} + L_3\, \widehat{k}\ ,$$

where $L_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}$, $L_2 = \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}$, $L_3 = \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}$.

Now $$\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{A}\right) = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ L_1 & L_2 & L_3 \end{vmatrix} = G_1\, \widehat{i} + G_2\, \widehat{j} + G_3\, \widehat{k}\ , \text{(say)}\ .$$

Here $$G_1 = \frac{\partial L_3}{\partial y} - \frac{\partial L_2}{\partial z} = \frac{\partial}{\partial y} \left\{ \frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y} \right\} - \frac{\partial}{\partial z} \left\{ \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} \right\}$$

$$= \frac{\partial^2 A_2}{\partial y \partial x} - \frac{\partial^2 A_1}{\partial y^2} - \frac{\partial^2 A_1}{\partial z^2} + \frac{\partial^2 A_3}{\partial z \partial x}$$

$$= \frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x} + \frac{\partial^2 A_1}{\partial x^2} - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2} \right)$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \right) - \vec{\nabla}^2 A_1 = \frac{\partial}{\partial x} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_1 \, .$$

Similarly, $G_2 = \frac{\partial}{\partial y} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_2$ and $G_3 = \frac{\partial}{\partial z} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_3$.

$$\therefore \quad \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \left\{ \frac{\partial}{\partial x} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_1 \right\} \hat{i} + \left\{ \frac{\partial}{\partial y} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_2 \right\} \hat{j}$$

$$+ \left\{ \frac{\partial}{\partial z} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 A_3 \right\} \hat{k}$$

$$= \left\{ \hat{i} \, \frac{\partial}{\partial x} + \hat{j} \, \frac{\partial}{\partial y} + \hat{k} \, \frac{\partial}{\partial z} \right\} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 \left(A_1 \, \hat{i} + A_2 \, \hat{j} + A_3 \, \hat{k} \right)$$

$$= \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \vec{\nabla}^2 \vec{A} \, . \textbf{ (Proved)}$$

 # Question :- Prove that for any differentiable scalar function ϕ, ϕ gradϕ is irrotational.

 # Answer :- Let $\vec{a} = \phi$ grad$\phi = \phi \, \vec{\nabla} \phi$.
The vector $\vec{a}$ is irrotational, if $\vec{\nabla} \times \vec{a} = \vec{0}$.

Now $\vec{\nabla} \times \vec{a} = \vec{\nabla} \times \left(\phi \, \vec{\nabla} \phi \right) = \phi \, \vec{\nabla} \times \left(\vec{\nabla} \phi \right) + \vec{\nabla} \phi \times \vec{\nabla} \phi = \phi \, \vec{\nabla} \times \left(\vec{\nabla} \phi \right)$. (1)

Again $$\vec{\nabla} \times \left(\vec{\nabla} \phi \right) = \left(\sum \hat{i} \, \frac{\partial}{\partial x} \right) \times \left(\vec{\nabla} \phi \right) = \sum \hat{i} \times \frac{\partial}{\partial x} \left\{ \hat{i} \, \frac{\partial \phi}{\partial x} + \hat{j} \, \frac{\partial \phi}{\partial y} + \hat{k} \, \frac{\partial \phi}{\partial z} \right\}$$

$$= \sum \hat{i} \times \left\{ \hat{i} \, \frac{\partial^2 \phi}{\partial x^2} + \hat{j} \, \frac{\partial^2 \phi}{\partial x \partial y} + \hat{k} \, \frac{\partial^2 \phi}{\partial x \partial z} \right\} = \sum \left\{ \hat{k} \, \frac{\partial^2 \phi}{\partial x \partial y} - \hat{j} \, \frac{\partial^2 \phi}{\partial x \partial z} \right\} = \vec{0} \, , \qquad (2)$$

using $\hat{i} \times \hat{i} = \vec{0}$, $\hat{i} \times \hat{j} = \hat{k}$, etc.; assuming continuous differentiability of ϕ will imply $\phi_{xy} = \phi_{yx}$, $\phi_{xz} = \phi_{zx}$, $\phi_{yz} = \phi_{zy}$; terms like $\hat{k} \, \frac{\partial^2 \phi}{\partial x \partial y}$ and $-\hat{k} \, \frac{\partial^2 \phi}{\partial y \partial x}$ will cancel.
From (1) and (2), we get $\vec{\nabla} \times \vec{a} = \vec{0}$, i.e., ϕ gradϕ is irrotational.

 # Question :- If $\vec{A} = 2xz^2 \, \hat{i} - yz \, \hat{j} + 3xz^3 \, \hat{k}$, $\phi = x^2 yz$, then find (i) curl$(\phi \vec{A})$, (ii) curl curl $\vec{A}$.

 # Answer :- (i) We know that

$$\text{curl}(\phi\vec{A}) = \vec{\nabla} \times (\phi\vec{A}) = \phi\left(\vec{\nabla} \times \vec{A}\right) + \vec{\nabla}\phi \times \vec{A}\ . \tag{1}$$

Now
$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix} = y\,\hat{i} + (4xz - 3z^3)\,\hat{j} + 0\,\hat{k}$$

and
$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\,\hat{i} + \frac{\partial\phi}{\partial y}\,\hat{j} + \frac{\partial\phi}{\partial z}\,\hat{k} = 2xyz\,\hat{i} + x^2z\,\hat{j} + x^2y\,\hat{k}\ .$$

$$\therefore\quad \phi\left(\vec{\nabla} \times \vec{A}\right) = x^2y^2z\,\hat{i} + (4x^3yz^2 - 3x^2yz^4)\,\hat{j} + 0\,\hat{k} \tag{2}$$

and
$$\vec{\nabla}\phi \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2xyz & x^2z & x^2y \\ 2xz^2 & -yz & 3xz^3 \end{vmatrix}$$

$$= (3x^3z^4 + x^2y^2z)\,\hat{i} + (2x^3yz^2 - 6x^2yz^4)\,\hat{j} + (-2xy^2z^2 - 2x^3z^3)\,\hat{k}\ . \tag{3}$$

From (1), (2) and (3), we have

$$\text{curl}(\phi\vec{A}) = (2x^2y^2z + 3x^3z^4)\,\hat{i} + (6x^3yz^2 - 9x^2yz^4)\,\hat{j} - (2xy^2z^2 + 2x^3z^3)\,\hat{k}\ .$$

(ii) We know that

$$\text{curl curl }(\vec{A}) = \vec{\nabla} \times \left(\vec{\nabla} \times \vec{A}\right) = \vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) - \vec{\nabla}^2\vec{A}\ . \tag{4}$$

Now
$$\vec{\nabla} \cdot \vec{A} = \frac{\partial}{\partial x}(2xz^2) + \frac{\partial}{\partial y}(-yz) + \frac{\partial}{\partial z}(3xz^3) = 2z^2 - z + 9xz^2,$$

$$\vec{\nabla}\left(\vec{\nabla} \cdot \vec{A}\right) = \frac{\partial}{\partial x}(2z^2 - z + 9xz^2)\,\hat{i} + \frac{\partial}{\partial y}(2z^2 - z + 9xz^2)\,\hat{j} + \frac{\partial}{\partial z}(2z^2 - z + 9xz^2)\,\hat{k}$$

$$= 9z^2\,\hat{i} + 0\,\hat{j} + (4z - 1 + 18xz)\,\hat{k} \tag{5}$$

and
$$\vec{\nabla}^2\vec{A} = \frac{\partial^2}{\partial x^2}\left(2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}\right) + \frac{\partial^2}{\partial y^2}\left(2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}\right)$$

$$+ \frac{\partial^2}{\partial z^2}\left(2xz^2\,\hat{i} - yz\,\hat{j} + 3xz^3\,\hat{k}\right) = \vec{0} + \vec{0} + 4x\,\hat{i} + 18xz\,\hat{k}\ . \tag{6}$$

From (4), (5) and (6), we have

$$\text{curl curl }(\vec{A}) = (9z^2 - 4x)\,\hat{i} + 0\,\hat{j} + (4z - 1)\,\hat{k}\ .$$

 # Question :- Prove that div(grad$f$) $= \vec{\nabla}^2 f$.

 # Answer :- div(grad$f$) $= \vec{\nabla}\cdot(\vec{\nabla}f)$
$= \left(\hat{i}\,\frac{\partial}{\partial x}+\hat{j}\,\frac{\partial}{\partial y}+\hat{k}\,\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\,\frac{\partial f}{\partial x}+\hat{j}\,\frac{\partial f}{\partial y}+\hat{k}\,\frac{\partial f}{\partial z}\right)$
$= \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2} = \vec{\nabla}^2 f.$ **(Proved)**

 # Question :- Prove that for the vector $\vec{r} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$, $\vec{\nabla}^2\left(\frac{1}{|\vec{r}|}\right) = 0.$

 # Answer :- $\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = \sum\hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\sum\hat{i}\,\frac{1}{r^2}\,\frac{\partial r}{\partial x} = -\frac{\sum x\,\hat{i}}{r^3}.$

$$\left[\ \because\ r^2 = x^2+y^2+z^2\ ,\ \Rightarrow\ \frac{\partial r}{\partial x} = \frac{x}{r}\ .\ \right]$$

Now
$$\vec{\nabla}^2\left(\frac{1}{|\vec{r}|}\right) = \vec{\nabla}\cdot\vec{\nabla}\left(\frac{1}{|\vec{r}|}\right) = \sum\hat{i}\,\frac{\partial}{\partial x}\cdot\left(-\frac{\sum x\,\hat{i}}{r^3}\right) = -\sum\frac{\partial}{\partial x}\left\{\frac{x}{r^3}\right\}$$
$$= -\sum\left\{\frac{r^3-3r^2\,\frac{\partial r}{\partial x}\,x}{r^6}\right\} = -\frac{1}{r^6}\ \sum\{r^3-3rx^2\}$$
$$= -\frac{1}{r^6}\ \left\{3r^3-3r\sum x^2\right\} = -\frac{1}{r^6}\ \left\{3r^3-3r^3\right\} = 0\ .\ (\textbf{Proved})$$

 # Question :- Find the gradient and Laplacian of the function $\phi = \sin(kx)\ \sin(ly)\ e^{\sqrt{k^2+l^2}\ z}$.

 # Answer :- The gradient of a scalar point function $\phi(x,y,z)$ is

$$\vec{\nabla}\phi = \text{grad}\ \phi = \sum\hat{i}\,\frac{\partial\phi}{\partial x}\ , \tag{1}$$

and the Laplacian of the function ϕ is

$$\vec{\nabla}^2\phi = \left(\vec{\nabla}\cdot\vec{\nabla}\right)\phi = \vec{\nabla}\cdot\left(\vec{\nabla}\phi\right) = \sum\frac{\partial^2\phi}{\partial x^2}\ . \tag{2}$$

Now
$$\frac{\partial\phi}{\partial x} = k\cos(kx)\sin(ly)\,e^{\sqrt{k^2+l^2}\ z}\ ,\ \frac{\partial^2\phi}{\partial x^2} = -k^2\sin(kx)\sin(ly)\,e^{\sqrt{k^2+l^2}\ z}\ ,$$
$$\frac{\partial\phi}{\partial y} = l\sin(kx)\cos(ly)\ e^{\sqrt{k^2+l^2}\ z}\ ,\ \frac{\partial^2\phi}{\partial y^2} = -l^2\sin(kx)\sin(ly)\ e^{\sqrt{k^2+l^2}\ z}\ ,$$
$$\frac{\partial\phi}{\partial z} = \sqrt{k^2+l^2}\sin(kx)\sin(ly)\ e^{\sqrt{k^2+l^2}\ z}\ ,\ \frac{\partial^2\phi}{\partial z^2} = (k^2+l^2)\sin(kx)\sin(ly)\ e^{\sqrt{k^2+l^2}\ z}\ .$$
$$\therefore\ \ \vec{\nabla}\phi = e^{\sqrt{k^2+l^2}\ z}\ [k\cos(kx)\sin(ly)\,\hat{i}+l\sin(kx)\cos(ly)\,\hat{j}+\sqrt{k^2+l^2}\sin(kx)\sin(ly)\,\hat{k}\ ]$$

and $\quad \overrightarrow{\nabla}^2\phi = [-k^2 - l^2 + (k^2+l^2)]\sin(kx)\ \sin(ly)\ e^{\sqrt{k^2+l^2}\ z} = 0\ .$

$\overrightarrow{\nabla}^2\phi = 0$ is called Laplace equation.

 # Question :- Find grad$(\log|\overrightarrow{r}|)$, where $\overrightarrow{r} = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$.

 # Answer :- Since $\overrightarrow{r} = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$.
Therefore $|\overrightarrow{r}| = \sqrt{x^2+y^2+z^2}$ and $\log|\overrightarrow{r}| = \frac{1}{2}\ \log(x^2+y^2+z^2)$.

$$\therefore\ \overrightarrow{\nabla}\left(\ \log|\overrightarrow{r}|\ \right) = \sum \widehat{i}\ \frac{\partial}{\partial x}\left(\ \log|\overrightarrow{r}|\ \right) = \frac{1}{2}\sum \widehat{i}\ \frac{\partial}{\partial x}\ \log(x^2+y^2+z^2)$$

$$= \frac{1}{2}\ \sum \widehat{i}\ \frac{2x}{x^2+y^2+z^2} = \frac{\sum x\,\widehat{i}}{x^2+y^2+z^2} = \frac{\overrightarrow{r}}{r^2}.$$

 # Question :- Find a simplified form of $\overrightarrow{\nabla}\times\{\overrightarrow{r}\ f(r)\}$, where $f(r)$ is differentiable.

 # Answer :- Here

$$\overrightarrow{\nabla}\times\{\overrightarrow{r}\ f(r)\} = f(r)\left(\overrightarrow{\nabla}\times\overrightarrow{r}\right) + \overrightarrow{\nabla}\left(f(r)\right)\times\overrightarrow{r}$$

$$= f(r)\left(\overrightarrow{\nabla}\times\overrightarrow{r}\right) + \frac{f'(r)}{r}\ (\overrightarrow{r}\times\overrightarrow{r}) = \overrightarrow{0}\ .$$

$$\left[\ \because\ \overrightarrow{\nabla}\times\overrightarrow{r} = \overrightarrow{0}\ ,\ \overrightarrow{r}\times\overrightarrow{r} = \overrightarrow{0}\ \right]$$

Thus the vector field $f(r)\ \overrightarrow{r}$ is irrotational, provided $f(r)$ is differentiable.

 # Question :- Prove $\nabla^2 f(r) = \frac{d^2f}{dr^2} + \frac{2}{r}\ \frac{df}{dr}$, where $r = \sqrt{x^2+y^2+z^2}$. Find $f(r)$ such that $\nabla^2 f(r) = 0$.

 # Answer :- Here

$$\overrightarrow{\nabla}^2 f(r) = \overrightarrow{\nabla}\cdot\left\{\overrightarrow{\nabla} f(r)\right\} = \overrightarrow{\nabla}\cdot\left\{\frac{f'(r)}{r}\ \overrightarrow{r}\right\}$$

$$= \frac{f'(r)}{r}\left(\overrightarrow{\nabla}\cdot\overrightarrow{r}\right) + \overrightarrow{r}\cdot\left\{\overrightarrow{\nabla}\left(\frac{f'(r)}{r}\right)\right\}$$

$$= \frac{3f'(r)}{r} + \overrightarrow{r}\cdot\left\{-f'(r)\ \frac{\overrightarrow{r}}{r^3} + \frac{1}{r^2}\ f''(r)\overrightarrow{r}\right\}$$

$$= \frac{3f'(r)}{r} + \frac{f''(r)}{r^2}(\overrightarrow{r}\cdot\overrightarrow{r}) - \frac{f'(r)}{r^3}\ (\overrightarrow{r}\cdot\overrightarrow{r})$$

$$= \frac{3f'(r)}{r} + f''(r) - \frac{f'(r)}{r} = \frac{d^2 f}{dr^2} + \frac{2}{r}\,\frac{df}{dr} \;.\; (\textbf{Proved})$$

Now $\vec{\nabla}^2 f(r) = 0 \;,\; \Rightarrow \frac{d^2 f}{dr^2} + \frac{2}{r}\,\frac{df}{dr} = 0 \;,\; \Rightarrow \frac{dz}{dr} + \frac{2}{r}\,z = 0 \;,\; \left[\text{put } \frac{df}{dr} = z\right]$

$$\Rightarrow \frac{dz}{z} + 2\,\frac{dr}{r} = 0 \;.$$

On integrating, we get

$$\log z + 2\log r = \log c_1 \;,\; \text{ where } c_1 \text{ be an integrating constant}$$

$$\text{or, } zr^2 = c_1 \quad \text{or, } \frac{df}{dr} = \frac{c_1}{r^2} \quad \text{or, } df = c_1\,\frac{dr}{r^2} \;.$$

On integrating, we get

$$f = c_2 - \frac{c_1}{r} \;,\; \text{ where } c_2 \text{ be an integrating constant} \;,$$

which is the required function $f(r)$.

 # Question :- Show that $\vec{\nabla}\cdot\left\{\frac{f(r)}{r}\,\vec{r}\right\} = \frac{1}{r^2}\,\frac{d}{dr}\{r^2 f(r)\}$. Hence show that $r^n\vec{r}$ is solenoidal if $n+3=0$.

 # Answer :-

$$\vec{\nabla}\cdot\left\{\frac{f(r)}{r}\,\vec{r}\right\} = \frac{f(r)}{r}\left(\vec{\nabla}\cdot\vec{r}\right) + \vec{r}\cdot\left\{\vec{\nabla}\left(\frac{f(r)}{r}\right)\right\} \;. \tag{1}$$

$$\text{Now} \qquad \vec{\nabla}\cdot\vec{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \tag{2}$$

$$\text{and} \quad \vec{\nabla}\left(\frac{f(r)}{r}\right) = \sum \hat{i}\,\frac{\partial}{\partial x}\left(\frac{f(r)}{r}\right) = \sum \hat{i}\left\{\frac{1}{r}\,f'(r)\,\frac{\partial r}{\partial x} - \frac{1}{r^2}\,f(r)\,\frac{\partial r}{\partial x}\right\}$$

$$= \frac{f'(r)}{r^2}\,\vec{r} - \frac{f(r)}{r^3}\,\vec{r} \;.\; \left[\because r^2 = x^2+y^2+z^2 \;,\; \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}\right] \tag{3}$$

From (1), (2) and (3), we get

$$\vec{\nabla}\cdot\left\{\frac{f(r)}{r}\,\vec{r}\right\} = \frac{3f(r)}{r} + \vec{r}\cdot\left\{\frac{f'(r)}{r^2}\,\vec{r} - \frac{f(r)}{r^3}\,\vec{r}\right\}$$

$$= \frac{3f(r)}{r} + f'(r) - \frac{f(r)}{r} = \frac{2}{r}\,f(r) + \frac{df}{dr} = \frac{1}{r^2}\,\frac{d}{dr}\{r^2 f(r)\} \;.\; (\textbf{Proved})$$

If we take $f(r) = r^{n+1}$, then $\vec{\nabla} \cdot \{r^n \vec{r}\} = \frac{1}{r^2} \frac{d}{dr}\{r^{n+3}\}$.
If $r^n \vec{r}$ is solenoidal vector, then $\vec{\nabla} \cdot \{r^n \vec{r}\} = 0$.
It is possible when $\frac{d}{dr}\{r^{n+3}\} = 0$, i.e., $n + 3 = 0$.

 # Question :- Find divergence and curl of the vector $\vec{v} = \frac{\hat{r}}{r}$, where $\hat{r}$ is the unit vector along $\vec{r}$ and r is the magnitude of the vector $\vec{r}$, where $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$.

 # Answer :- Let $\vec{v} = \frac{\hat{r}}{r}$, where $\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$, $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $r^2 = x^2 + y^2 + z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc. $\therefore \quad \vec{v} = \frac{\vec{r}}{r^2}$.

Now
$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}\left(\vec{\nabla} \cdot \vec{r}\right) + \vec{r} \cdot \vec{\nabla}\left(\frac{1}{r^2}\right)$$
$$= \frac{3}{r^2} + \vec{r} \cdot \left(-\frac{2\,\vec{r}}{r^4}\right) = \frac{3}{r^2} - \frac{2r^2}{r^4} = \frac{1}{r^2} \, ,$$
$$\left[\because \vec{\nabla}\left(\frac{1}{r^2}\right) = \sum \hat{i}\, \frac{\partial}{\partial x}\left(\frac{1}{r^2}\right) = -\sum \hat{i}\, \frac{2}{r^3}\, \frac{\partial r}{\partial x} = -2\, \frac{\sum x\,\hat{i}}{r^4} = -\frac{2\,\vec{r}}{r^4} \right]$$

and
$$\vec{\nabla} \times \vec{v} = \vec{\nabla} \times \left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}\left(\vec{\nabla} \times \vec{r}\right) + \vec{\nabla}\left(\frac{1}{r^2}\right) \times \vec{r}$$
$$= \frac{1}{r^2}\left(\vec{\nabla} \times \vec{r}\right) - \frac{2}{r^4}\left(\vec{r} \times \vec{r}\right) = \vec{0} \, .$$
$$\left[\because \ \vec{r} \times \vec{r} = \vec{0} \ \& \ \vec{\nabla} \times \vec{r} = \vec{0} \right]$$

 # Question :- Prove that for the vector $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$, $\frac{\vec{r}}{|\vec{r}|^3}$ is both solenoidal and irrotational.

 # Answer :- Let $\vec{v} = \frac{\vec{r}}{|\vec{r}|^3}$, where $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $r^2 = x^2 + y^2 + z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Now
$$\vec{\nabla} \cdot \vec{v} = \vec{\nabla} \cdot \left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left(\vec{\nabla} \cdot \vec{r}\right) + \vec{r} \cdot \vec{\nabla}\left(\frac{1}{r^3}\right)$$
$$= \frac{3}{r^3} + \vec{r} \cdot \left(-\frac{3\,\vec{r}}{r^5}\right) = \frac{3}{r^3} - \frac{3r^2}{r^5} = 0 \, .$$
$$\left[\because \vec{\nabla}\left(\frac{1}{r^3}\right) = \sum \hat{i}\, \frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = -\sum \hat{i}\, \frac{3}{r^4}\, \frac{\partial r}{\partial x} = -3\, \frac{\sum x\,\hat{i}}{r^5} = -\frac{3\,\vec{r}}{r^5} \right]$$

Hence $\vec{v} = \frac{\vec{r}}{|\vec{r}|^3}$ is solenoidal vector .

Again $$\vec{\nabla}\times\vec{v} = \vec{\nabla}\times\left(\frac{\vec{r}}{r^3}\right) = \frac{1}{r^3}\left(\vec{\nabla}\times\vec{r}\right)+\vec{\nabla}\left(\frac{1}{r^3}\right)\times\vec{r}$$

$$= \frac{1}{r^3}\left(\vec{\nabla}\times\vec{r}\right) - \frac{3}{r^5}\left(\vec{r}\times\vec{r}\right) = \vec{0}\ .$$

$$\left[\ \because\ \vec{r}\times\vec{r} = \vec{0}\ \&\ \vec{\nabla}\times\vec{r} = \vec{0}\ \right]$$

Hence $\vec{v} = \frac{\vec{r}}{|\vec{r}|^3}$ is an irrotational vector.

 # Question :- Show that $r^n\ \vec{r}$ is an irrotational vector for any value of n but is solenoidal, if $n+3=0$.

 # Answer :- Let $\vec{A} = r^n\ \vec{r}$, where $\vec{r} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$ and $r^2 = x^2+y^2+z^2$, $\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

Now $$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times\left(r^n\ \vec{r}\right) = r^n\left(\vec{\nabla}\times\vec{r}\right) + \vec{\nabla}\left(r^n\right)\times\vec{r}$$

$$= r^n\left(\vec{\nabla}\times\vec{r}\right) + nr^{n-2}\left(\vec{r}\times\vec{r}\right) = \vec{0}\ .$$

$$\left[\ \because\ \vec{r}\times\vec{r} = \vec{0}\ \&\ \vec{\nabla}\times\vec{r} = \vec{0}\ \right]$$

Hence $\vec{A} = r^n\ \vec{r}$ is an irrotational vector for any value of n.

Again $$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot\left(r^n\ \vec{r}\right) = r^n\left(\vec{\nabla}\cdot\vec{r}\right) + \vec{r}\cdot\vec{\nabla}\left(r^n\right)$$

$$= 3r^n + \vec{r}\cdot\left(nr^{n-2}\vec{r}\right) = 3r^n + nr^n = (3+n)\ r^n\ .$$

Hence, if $\vec{\nabla}\cdot\vec{A} = 0$, then $n+3=0$. Consequently $r^n\ \vec{r}$ is solenoidal vector, if $n+3=0$.

 # Question :- If $\vec{r} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$ and $r = |\vec{r}|$, then prove that $\vec{\nabla}\cdot\left[r\ \vec{\nabla}\left(\frac{1}{r^3}\right)\right] = 3r^{-4}$.

 # Answer :- Here $\vec{r} = x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$ and $r^2 = x^2+y^2+z^2$, $\frac{\partial r}{\partial x} = \frac{x}{r}$, etc.

$$\therefore\ \vec{\nabla}\left(\frac{1}{r^3}\right) = \sum\hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = -\sum\hat{i}\,\frac{3}{r^4}\,\frac{\partial r}{\partial x} = -3\,\frac{\sum x\,\hat{i}}{r^5} = -\frac{3\vec{r}}{r^5}\ .$$

Let $\vec{v} = r\,\vec{\nabla}\left(\frac{1}{r^3}\right) = -3\,\frac{\vec{r}}{r^4}$.

Now $$\vec{\nabla}\cdot\vec{v} = -3\,\vec{\nabla}\cdot\left(\frac{\vec{r}}{r^4}\right) = -3\left\{\frac{1}{r^4}\left(\vec{\nabla}\cdot\vec{r}\right) + \vec{r}\cdot\vec{\nabla}\left(\frac{1}{r^4}\right)\right\}$$

$$= -3\left\{\frac{3}{r^4} + \vec{r}\cdot\left(-\frac{4\vec{r}}{r^6}\right)\right\} = -3\left\{\frac{3}{r^4} - \frac{4}{r^4}\right\} = 3r^{-4}\ . \ \textbf{(Proved)}$$

$$\left[\ \because\ \vec{\nabla}\left(\frac{1}{r^4}\right) = \sum \widehat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r^4}\right) = -\sum \widehat{i}\,\frac{4}{r^5}\,\frac{\partial r}{\partial x} = -4\,\frac{\sum x\,\widehat{i}}{r^6} = -\frac{4\vec{r}}{r^6}\right]$$

 # Question :- Show that $\vec{\nabla}\left(\vec{A}\cdot\vec{\nabla}\frac{1}{r}\right) = \frac{3\,(\vec{A}\cdot\vec{r})}{r^5}\,\vec{r} - \frac{\vec{A}}{r^3}$, where $r^2 = x^2 + y^2 + z^2$, $\vec{r} = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$, $\vec{A}$ is a constant vector and $\widehat{i}$, $\widehat{j}$, $\widehat{k}$ are orthogonal unit vectors.

 # Answer :- $\vec{\nabla}\left(\frac{1}{r}\right) = \sum \widehat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right) = -\sum \widehat{i}\,\frac{1}{r^2}\,\frac{\partial r}{\partial x} = -\frac{\sum x\,\widehat{i}}{r^3} = -\frac{\vec{r}}{r^3}$.

$$\therefore\ \vec{A}\cdot\vec{\nabla}\left(\frac{1}{r}\right) = \vec{A}\cdot\left(-\frac{\vec{r}}{r^3}\right) = -\frac{\vec{A}\cdot\vec{r}}{r^3}\ .$$

Hence $$\vec{\nabla}\left(\vec{A}\cdot\vec{\nabla}\frac{1}{r}\right) = \vec{\nabla}\left(-\frac{\vec{A}\cdot\vec{r}}{r^3}\right) = -\sum \widehat{i}\,\frac{\partial}{\partial x}\left(\frac{\vec{A}\cdot\vec{r}}{r^3}\right)$$

$$= -\left\{-\sum \widehat{i}\,\frac{3\left(\vec{A}\cdot\vec{r}\right)}{r^4}\,\frac{\partial r}{\partial x} + \sum \widehat{i}\,\frac{1}{r^3}\,\frac{\partial}{\partial x}\left(\vec{A}\cdot\vec{r}\right)\right\}$$

$$= \frac{3}{r^5}\,\left(\vec{A}\cdot\vec{r}\right)\sum x\,\widehat{i} - \frac{1}{r^3}\sum \widehat{i}\,\left(\frac{\partial\vec{A}}{\partial x}\cdot\vec{r} + \frac{\partial\vec{r}}{\partial x}\cdot\vec{A}\right)$$

$$= \frac{3\,\left(\vec{A}\cdot\vec{r}\right)}{r^5}\,\vec{r} - \frac{1}{r^3}\sum \widehat{i}\,\left(\vec{0}\cdot\vec{r} + \widehat{i}\cdot\vec{A}\right)$$

[since $\vec{A}$ is constant vector, so $\frac{\partial\vec{A}}{\partial x} = \vec{0}$, $\frac{\partial\vec{r}}{\partial x} = \widehat{i}$ and $\vec{A}\cdot\widehat{i} = A_1$.]

$$= \frac{3\,\left(\vec{A}\cdot\vec{r}\right)}{r^5}\,\vec{r} - \frac{1}{r^3}\sum A_1\,\widehat{i} = \frac{3\,\left(\vec{A}\cdot\vec{r}\right)}{r^5}\,\vec{r} - \frac{\vec{A}}{r^3}\ .\ \textbf{(Proved)}$$

 # Question :- If $\vec{r} = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$, $r = |\vec{r}|$ and $\vec{a}$ is a constant vector, then show that
(a) curl $(\vec{a}\times\vec{r}) = 2\,\vec{a}$

(b) grad $(r^n) = nr^{n-2}\ \overrightarrow{r}$
(c) curl$\{r^n(\overrightarrow{a} \times \overrightarrow{r})\} = (n+2)\ r^n\overrightarrow{a} - nr^{n-2}(\overrightarrow{r}\cdot\overrightarrow{a})\overrightarrow{r}$.

 # Answer :- (a) curl $(\overrightarrow{a} \times \overrightarrow{r})$

$$= \sum \widehat{i}\frac{\partial}{\partial x} \times (\overrightarrow{a} \times \overrightarrow{r}) = \sum \widehat{i} \times \left\{\frac{\partial \overrightarrow{a}}{\partial x} \times \overrightarrow{r} + \overrightarrow{a} \times \frac{\partial \overrightarrow{r}}{\partial x}\right\}$$

$$= \sum \widehat{i} \times \left\{\overrightarrow{0} + \overrightarrow{a} \times \widehat{i}\right\}\ [\ \because\ \overrightarrow{a} \text{ is a constant vector}]$$

$$= \sum \widehat{i} \times \left(\overrightarrow{a} \times \widehat{i}\right) = \sum \left\{(\widehat{i}\cdot\widehat{i})\overrightarrow{a} - (\widehat{i}\cdot\overrightarrow{a})\widehat{i}\right\}$$

$$= 3\overrightarrow{a} - \overrightarrow{a} = 2\ \overrightarrow{a}\ .$$

$$\therefore\quad \text{curl } (\overrightarrow{a} \times \overrightarrow{r}) = 2\ \overrightarrow{a}\ . \tag{1}$$

(b) grad (r^n)

$$= \sum \widehat{i}\ \frac{\partial}{\partial x}(r^n) = \sum \widehat{i}\ n\ r^{n-1}\ \frac{\partial r}{\partial x} = \sum \widehat{i}\ n\ r^{n-2}\ x = n\ r^{n-2}\ \overrightarrow{r}\ . \tag{2}$$

(c) curl$\{r^n(\overrightarrow{a} \times \overrightarrow{r})\} = \overrightarrow{\nabla} \times \{r^n(\overrightarrow{a} \times \overrightarrow{r})\}$

$$= r^n\{\overrightarrow{\nabla} \times (\overrightarrow{a} \times \overrightarrow{r})\} + \overrightarrow{\nabla}(r^n) \times (\overrightarrow{a} \times \overrightarrow{r})$$

$$= r^n\ 2\overrightarrow{a} + n\ r^{n-2}\ \overrightarrow{r} \times (\overrightarrow{a} \times \overrightarrow{r})\ [\text{ by } (1), (2)]$$

$$= 2r^n\overrightarrow{a} + n\ r^{n-2}\{(\overrightarrow{r}\cdot\overrightarrow{r})\overrightarrow{a} - (\overrightarrow{r}\cdot\overrightarrow{a})\overrightarrow{r}\}$$

$$= 2r^n\overrightarrow{a} + n\ r^n\overrightarrow{a} - nr^{n-2}(\overrightarrow{r}\cdot\overrightarrow{a})\overrightarrow{r}$$

$$= (n+2)\ r^n\overrightarrow{a} - nr^{n-2}(\overrightarrow{r}\cdot\overrightarrow{a})\overrightarrow{r}\ .$$

 # Question :- If $\overrightarrow{a}$ be a constant vector, then prove that
(i) curl $(\overrightarrow{a}\cdot\overrightarrow{r})\overrightarrow{a} = \overrightarrow{0}$
(ii) curl $\left(\frac{\overrightarrow{a}\times\overrightarrow{r}}{r^3}\right) = -\frac{\overrightarrow{a}}{r^3} + \frac{3\ \overrightarrow{r}}{r^5}\ (\overrightarrow{a}\cdot\overrightarrow{r})$, $[\ r = |\overrightarrow{r}|\ ]$.

 # Answer :- Let $\overrightarrow{a} = a_1\ \widehat{i} + a_2\ \widehat{j} + a_3\ \widehat{k}$, where $a_1,\ a_2,\ a_3$ are constants and $\overrightarrow{r} = x\ \widehat{i} + y\ \widehat{j} + z\ \widehat{k}$, $r^2 = x^2 + y^2 + z^2$, $\overrightarrow{a}\cdot\overrightarrow{r} = a_1\ x + a_2\ y + a_3\ z$.

(i) Now curl $(\overrightarrow{a}\cdot\overrightarrow{r})\overrightarrow{a} = \overrightarrow{\nabla} \times \{(\overrightarrow{a}\cdot\overrightarrow{r})\ \overrightarrow{a}\}$

$$= (\overrightarrow{a}\cdot\overrightarrow{r})(\overrightarrow{\nabla} \times \overrightarrow{a}) + \overrightarrow{\nabla}(\overrightarrow{a}\cdot\overrightarrow{r}) \times \overrightarrow{a}\ . \tag{1}$$

Now $\overrightarrow{\nabla} \times \overrightarrow{a} = \overrightarrow{0}$ and $\overrightarrow{\nabla}(\overrightarrow{a}\cdot\overrightarrow{r}) = \sum \widehat{i}\ \frac{\partial}{\partial x}(a_1x + a_2y + a_3z) = \sum \widehat{i}\ a_1 = \overrightarrow{a}$.
From (1), we get

$$\text{curl } (\overrightarrow{a}\cdot\overrightarrow{r})\overrightarrow{a} = \overrightarrow{0} + \overrightarrow{a} \times \overrightarrow{a} = \overrightarrow{0}\ .\ \textbf{(Proved)}$$

(ii) curl $\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \vec{\nabla}\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right)$

$$= \frac{1}{r^3}\{\vec{\nabla}\times(\vec{a}\times\vec{r})\} + \vec{\nabla}\left(\frac{1}{r^3}\right)\times(\vec{a}\times\vec{r})\ . \qquad (1)$$

Now curl $(\vec{a}\times\vec{r}) = \vec{\nabla}\times(\vec{a}\times\vec{r})$

$$= \sum \hat{i}\frac{\partial}{\partial x}\times(\vec{a}\times\vec{r}) = \sum \hat{i}\times\left\{\frac{\partial\vec{a}}{\partial x}\times\vec{r} + \vec{a}\times\frac{\partial\vec{r}}{\partial x}\right\}$$

$$= \sum \hat{i}\times\left\{\vec{0} + \vec{a}\times\hat{i}\right\} \ [\ \because\ \vec{a} \text{ is a constant vector}]$$

$$= \sum \hat{i}\times\left(\vec{a}\times\hat{i}\right) = \sum\left\{(\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\hat{i}\right\}$$

$$= 3\vec{a} - \vec{a} = 2\,\vec{a}\ , \qquad (2)$$

and grad $\left(\frac{1}{r^3}\right) = \vec{\nabla}\left(\frac{1}{r^3}\right)$

$$= \sum \hat{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3}\right) = -3\sum \hat{i}\,\frac{1}{r^4}\,\frac{\partial r}{\partial x} = -3\sum \hat{i}\,\frac{x}{r^5} = -\frac{3\,\vec{r}}{r^5}\ . \qquad (3)$$

From (1), (2) and (3), we get

$$\text{curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{2\vec{a}}{r^3} - \frac{3\,\vec{r}}{r^5}\times(\vec{a}\times\vec{r})$$

$$= \frac{2\vec{a}}{r^3} - \frac{3}{r^5}\{(\vec{r}\cdot\vec{r})\vec{a} - (\vec{r}\cdot\vec{a})\vec{r}\}$$

$$= \frac{2\vec{a}}{r^3} - \frac{3\vec{a}}{r^3} + \frac{3(\vec{r}\cdot\vec{a})\vec{r}}{r^5}$$

$$= -\frac{\vec{a}}{r^3} + \frac{3}{r^5}\,(\vec{r}\cdot\vec{a})\,\vec{r}\ .\ (\textbf{Proved})$$

 # Question :- If the vector $\vec{f} = 3x\,\hat{i} + (x+y)\,\hat{j} - az\,\hat{k}$ is solenoidal, find a.

 # Answer :- The vector $\vec{f}$ is solenoidal, so $\vec{\nabla}\cdot\vec{f} = 0$.

Now $\quad \vec{\nabla}\cdot\vec{f} = \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(-az) = 3+1-a = 4-a\ .$

Since $\vec{\nabla}\cdot\vec{f} = 0$, so $a = 4$. Hence the required value of 'a' is 4.

 # Question :- For what value of the constant a, will the vector field $\vec{A} = (axy - z^3)\,\widehat{i} + (a-2)x^2\,\widehat{j} + (1-a)xz^2\,\widehat{k}$ be always irrotational?

 # Answer :- The vector $\vec{A}$ is always irrotational, so $\vec{\nabla} \times \vec{A} = \vec{0}$.

Now $$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix}$$

$$= (0-0)\,\widehat{i} + \left\{-3z^2 - (1-a)z^2\right\}\,\widehat{j} + \{2(a-2)x - ax\}\,\widehat{k}$$

$$= 0\,\widehat{i} + (a-4)z^2\,\widehat{j} + (a-4)x\,\widehat{k}\,.$$

Since $\vec{\nabla} \times \vec{A} = \vec{0}$, so $a = 4$.
Hence the value of a is 4.

 # Question :- Find the constants $a,\ b,\ c$ so that $\vec{f} = (x + 2y + az)\,\widehat{i} + (bx - 3y - z)\,\widehat{j} + (4x + cy + 2z)\,\widehat{k}$ is irrotational.

 # Answer :- The vector $\vec{f}$ is irrotational, so $\vec{\nabla} \times \vec{f} = \vec{0}$.

Now $$\vec{\nabla}\times\vec{f} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + 2y + az & bx - 3y - z & 4x + cy + 2z \end{vmatrix}$$

$$= (c+1)\,\widehat{i} + (a-4)\,\widehat{j} + (b-2)\,\widehat{k}\,.$$

Since $\vec{\nabla} \times \vec{f} = \vec{0}$, so $a = 4,\ b = 2,\ c = -1$.

 # Question :- Find the constants $a,\ b,\ c$ such that the vector $\vec{V} = (-4x - 3y + az)\,\widehat{i} + (bx + 3y + 5z)\,\widehat{j} + (4x + cy + 3z)\,\widehat{k}$ is irrotational.

 # Answer :- The vector $\vec{V}$ is irrotational, so $\vec{\nabla} \times \vec{V} = \vec{0}$.

Now $$\vec{\nabla}\times\vec{V} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -4x - 3y + az & bx + 3y + 5z & 4x + cy + 3z \end{vmatrix}$$

$$= (c-5)\,\widehat{i} + (a-4)\,\widehat{j} + (b+3)\,\widehat{k}\,.$$

Since $\vec{\nabla} \times \vec{V} = \vec{0}$, so $a = 4,\ b = -3,\ c = 5$.

 # Question :- Show that the force field given by $\overrightarrow{F} = (4xy - 3x^2z^2)\,\widehat{i} + 2x^2\widehat{j} - 2x^3z\,\widehat{k}$ is irrotational. Find a scalar function ϕ such that $\overrightarrow{F} = \overrightarrow{\nabla}\phi$.

 # Answer :- We know that, the force field $\overrightarrow{F}$ be irrotational, iff $\overrightarrow{\nabla} \times \overrightarrow{F} = \overrightarrow{0}$.

Now $$\overrightarrow{\nabla} \times \overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= (0-0)\,\widehat{i} + (-6x^2z + 6x^2z)\,\widehat{j} + (4x - 4x)\,\widehat{k} = \overrightarrow{0}\ .$$

Hence $\overrightarrow{F}$ is an irrotational force field.
To find the scalar function ϕ of $\overrightarrow{F}$, we have

$$d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz$$

$$= \left\{\frac{\partial\phi}{\partial x}\,\widehat{i} + \frac{\partial\phi}{\partial y}\,\widehat{j} + \frac{\partial\phi}{\partial z}\,\widehat{k}\right\} \cdot \left\{dx\,\widehat{i} + dy\,\widehat{j} + dz\,\widehat{k}\right\} = \overrightarrow{\nabla}\phi \cdot d\overrightarrow{r}$$

$$= \overrightarrow{F} \cdot d\overrightarrow{r} = (4xy - 3x^2z^2)\,dx + 2x^2\,dy - 2x^3z\,dz$$

$$= 2(2xydx + x^2dy) - (3x^2z^2dx + 2x^3z\,dz) = d(2x^2y - x^3z^2)\ .$$

Integrating, we get

$$\phi = 2x^2y - x^3z^2 + c\ ,\ \text{ where } c \text{ is a constant of integration.}$$

 # Question :- Show that the vector $\overrightarrow{F} = (2x - yz)\,\widehat{i} + (2y - zx)\,\widehat{j} + (2z - xy)\,\widehat{k}$ is irrotational. For this $\overrightarrow{F}$, find a scalar function ϕ such that $\overrightarrow{F} = \overrightarrow{\nabla}\phi$.

 # Answer :- The vector $\overrightarrow{F}$ is irrotational, if $\overrightarrow{\nabla} \times \overrightarrow{F} = \overrightarrow{0}$.

Now $$\overrightarrow{\nabla} \times \overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - yz & 2y - zx & 2z - xy \end{vmatrix}$$

$$= (-x + x)\,\widehat{i} + (-y + y)\,\widehat{j} + (-z + z)\,\widehat{k} = \overrightarrow{0}\ ;$$

is satisfied and hence the vector $\overrightarrow{F}$ is an irrotational vector.
To find the scalar function ϕ of $\overrightarrow{F}$, we have $d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = \overrightarrow{F} \cdot d\overrightarrow{r}$.

$$\therefore\ d\phi = (2x - yz)\,dx + (2y - zx)\,dy + (2z - xy)\,dz$$

$$= (2x\ dx + 2y\ dy + 2z\ dz) - (yz\ dx + zx\ dy + xy\ dz)$$
$$= d(x^2 + y^2 + z^2) - d(xyz) = d(x^2 + y^2 + z^2 - xyz)\ .$$

Integrating, we get

$$\phi = x^2 + y^2 + z^2 - xyz + c\ ,\ \text{where } c \text{ is a constant of integration.}$$

 # Question :- Show that $\overrightarrow{A} = (6xy + z^3)\,\widehat{i} + (3x^2 - z)\,\widehat{j} + (3xz^2 - y)\,\widehat{k}$ is irrotational. Find ϕ such that $\overrightarrow{A} = \overrightarrow{\nabla}\phi$ and $\phi(1, 1, 1) = 3$.

 # Answer :- The vector $\overrightarrow{A}$ is irrotational, if $\overrightarrow{\nabla} \times \overrightarrow{A} = \overrightarrow{0}$.

Now $$\overrightarrow{\nabla}\times\overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6xy + z^3 & 3x^2 - z & 3xz^2 - y \end{vmatrix}$$

$$= (-1 + 1)\ \widehat{i} + \left(3z^2 - 3z^2\right)\ \widehat{j} + (6x - 6x)\ \widehat{k} = \overrightarrow{0}\ ;$$

is satisfied and hence the vector $\overrightarrow{F}$ is an irrotational vector.
To find the scalar function ϕ of $\overrightarrow{F}$, we have $d\phi = \frac{\partial\phi}{\partial x}\ dx + \frac{\partial\phi}{\partial y}\ dy + \frac{\partial\phi}{\partial z}\ dz = \overrightarrow{F} \cdot d\overrightarrow{r}$.

$$\therefore\ d\phi = (6xy + z^3)\ dx + (3x^2 - z)\ dy + (3xz^2 - y)\ dz$$
$$= (z^3dx + 3xz^2dz) + 3(2xy\ dx + x^2\ dy) - (z\ dy + y\ dz)$$
$$= d(xz^3 + 3x^2y - yz)\ .$$

Integrating, we get

$$\phi = xz^3 + 3x^2y - yz + c\ ,\ \text{where } c \text{ is a constant of integration .}$$

$$\because\ \phi(1, 1, 1) = 3\ ,\ \Rightarrow\ 3 = 1 + 3 - 1 + c\ ,\ \Rightarrow\ c = 0\ .$$

$\therefore\quad \phi = xz^3 + 3x^2y - yz.$

 # Question :- Show that the vector $\overrightarrow{F} = (y^2\cos x + z^3)\,\widehat{i} + (2y\sin x - 4)\,\widehat{j} + (3xz^2 + 2)\,\widehat{k}$ is irrotational. Find the scalar function ϕ for $\overrightarrow{F}$ such that $\overrightarrow{F} = \overrightarrow{\nabla}\phi$.

 # Answer :- The vector $\overrightarrow{F}$ is irrotational, if $\overrightarrow{\nabla} \times \overrightarrow{F} = \overrightarrow{0}$.

Now $$\overrightarrow{\nabla}\times\overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= (0-0)\,\widehat{i} - (3z^2 - 3z^2)\,\widehat{j} + (2y\cos x - 2y\cos x)\,\widehat{k} = \overrightarrow{0}\ ;$$

is satisfied and hence the vector $\overrightarrow{F}$ is an irrotational vector.
To find the scalar function ϕ of $\overrightarrow{F}$, we have $d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = \overrightarrow{F}\cdot d\overrightarrow{r}$.

$$\therefore\ d\phi = (y^2\cos x + z^3)\,dx + (2y\sin x - 4)\,dy + (3xz^2 + 2)\,dz$$

$$= y^2\cos x\,dx + 2y\sin x\,dy + z^3\,dx + 3xz^2\,dz - 4\,dy + 2\,dz$$
$$= d(y^2\sin x) + d(z^3x) - 4\,dy + 2\,dz = d(y^2\sin x + xz^3 - 4y + 2z)\ .$$

Integrating, we get

$$\phi = y^2\sin x + xz^3 - 4y + 2z + c\ ,\ \text{where } c \text{ is a constant of integration.}$$

 # Question :- Show that the vector $\overrightarrow{F} = (2xy + z^3)\,\widehat{i} + x^2\,\widehat{j} + 3xz^2\,\widehat{k}$ is irrotational. Find the scalar function ϕ for $\overrightarrow{F}$ such that $\overrightarrow{F} = \overrightarrow{\nabla}\phi$.

 # Answer :- The vector $\overrightarrow{F}$ is irrotational, if $\overrightarrow{\nabla}\times\overrightarrow{F} = \overrightarrow{0}$.

Now $$\overrightarrow{\nabla}\times\overrightarrow{F} = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= (0-0)\,\widehat{i} - (3z^2 - 3z^2)\,\widehat{j} + (2x - 2x)\,\widehat{k} = \overrightarrow{0}\ ;$$

is satisfied and hence the vector $\overrightarrow{F}$ is an irrotational vector.
To find the scalar function ϕ of $\overrightarrow{F}$, we have $d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = \overrightarrow{F}\cdot d\overrightarrow{r}$.

$$\therefore\ d\phi = (2xy + z^3)\,dx + x^2\,dy + 3xz^2\,dz$$

$$= (2xy\,dx + x^2\,dy) + (z^3\,dx + 3xz^2\,dz)$$
$$= d(x^2y) + d(xz^3) = d(x^2y + xz^3)\ .$$

Integrating, we get

$$\phi = x^2y + xz^3 + c\ ,\ \text{where } c \text{ is a constant of integration.}$$

 # Question :- Find the directional derivative of the function $f = x^2 - y^2 + z^2$ at the point $P(1, 2, -3)$ in the direction of the vector $\overrightarrow{PQ}$, where Q is the point $(3, 1, 2)$.

 # Answer :- The unit vector in the direction of the vector $\overrightarrow{PQ} = 2\widehat{i} - \widehat{j} + 5\widehat{k}$ is $\widehat{a} = \frac{1}{\sqrt{30}}\,(2\widehat{i} - \widehat{j} + 5\widehat{k})$.

Now $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial z}\,\widehat{k} = 2x\,\widehat{i} - 2y\,\widehat{j} + 2z\,\widehat{k}$.

$\therefore \quad \left[\vec{\nabla} f\right]_{(1,2,-3)} = 2\,\widehat{i} - 4\,\widehat{j} - 6\,\widehat{k}$.

Hence the directional derivative of f at the point $P(1,2,-3)$ in the direction of the vector $\overrightarrow{PQ}$ is

$$\vec{\nabla} f(1,2,-3)\cdot\widehat{a} = \frac{1}{\sqrt{30}}\,(4+4-30) = -\frac{22}{\sqrt{30}} = -\frac{11}{15}\,\sqrt{30}\,.$$

 # Question :- Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ, where Q is $(5,0,4)$.

 # Answer :- The unit vector in the direction of the line PQ is $\overrightarrow{PQ} = 4\widehat{i} - 2\widehat{j} + \widehat{k}$ is $\widehat{a} = \frac{1}{\sqrt{21}}\,(4\widehat{i} - 2\widehat{j} + \widehat{k})$.

Now $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial z}\,\widehat{k} = 2x\,\widehat{i} - 2y\,\widehat{j} + 4z\,\widehat{k}$.

$\therefore \quad \left[\vec{\nabla} f\right]_{(1,2,3)} = 2\,\widehat{i} - 4\,\widehat{j} + 12\,\widehat{k}$.

Hence the directional derivative of f at the point $P(1,2,3)$ in the direction of the line PQ is

$$\vec{\nabla} f(1,2,3)\cdot\widehat{a} = \frac{1}{\sqrt{21}}\,(8+8+12) = \frac{28}{\sqrt{21}}\,.$$

 # Question :- If $f = x^3 + y^3 + z^3$, find the directional derivative of f at the point $(1,-1,2)$ in the direction of the vector $\widehat{j} - \widehat{k}$.

 # Answer :- The unit vector in the direction of the vector $\widehat{j} - \widehat{k}$ is $\widehat{a} = \frac{1}{\sqrt{2}}\,(\widehat{j} - \widehat{k})$.

Now $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial z}\,\widehat{k} = 3x^2\,\widehat{i} + 3y^2\,\widehat{j} + 3z^2\,\widehat{k}$.

$\therefore \quad \left[\vec{\nabla} f\right]_{(1,-1,2)} = 3\,\widehat{i} + 3\,\widehat{j} + 12\,\widehat{k}$.

Hence the directional derivative of f at the point $(1,-1,2)$ in the direction of the vector $(\widehat{j} - \widehat{\widehat{k}})$ is

$$\vec{\nabla} f(1,-1,2)\cdot\widehat{a} = \left(3\,\widehat{i} + 3\,\widehat{j} + 12\,\widehat{k}\right)\cdot\left(0\,\widehat{i} + \frac{1}{\sqrt{2}}\,\widehat{j} - \frac{1}{\sqrt{2}}\,\widehat{k}\right)$$

$$= \frac{3}{\sqrt{2}} - \frac{12}{\sqrt{2}} = -\frac{9}{\sqrt{2}} = -\frac{9}{2}\,\sqrt{2}\,.$$

 # Question :- Find the directional derivative of the function $f(x,y,z) = yz + zx + xy$ in the direction $\vec{u} = \widehat{i} + 2\,\widehat{j} + 2\,\widehat{k}$ at the point $(1,2,0)$.

 # Answer :- The unit vector in the direction of the vector $\vec{u} = \hat{i} + 2\,\hat{j} + 2\,\hat{k}$ is $\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{1}{3}\,(\hat{i} + 2\,\hat{j} + 2\,\hat{k})$.

Now $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k} = (y+z)\,\hat{i} + (z+x)\,\hat{j} + (x+y)\,\hat{k}$.

$\therefore \quad \left[\vec{\nabla} f\right]_{(1,\ 2,\ 0)} = 2\,\hat{i} + \hat{j} + 3\,\hat{k}$.

Hence the directional derivative of f at the point $(1, 2, 0)$ in the direction of the vector $(\hat{i} + 2\,\hat{j} + 2\,\hat{k})$ is

$\vec{\nabla} f(1,2,0) \cdot \hat{u} = \left(2\,\hat{i} + \hat{j} + 3\,\hat{k}\right) \cdot \left(\frac{1}{3}\,\hat{i} + \frac{2}{3}\,\hat{j} + \frac{2}{3}\,\hat{k}\right) = \frac{2}{3} + \frac{2}{3} + \frac{6}{3} = \frac{10}{3}$.

 # Question :- Find the directional derivative of $\phi = xy^2z + 4x^2z$ at $(-1, 1, 2)$ in the direction $2\,\hat{i} + \hat{j} - 2\,\hat{k}$.

 # Answer :- The unit vector in the direction of the vector $2\,\hat{i} + \hat{j} - 2\,\hat{k}$ is $\hat{a} = \frac{1}{3}\,(2\,\hat{i} + \hat{j} - 2\,\hat{k})$.

Now $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\,\hat{i} + \frac{\partial \phi}{\partial y}\,\hat{j} + \frac{\partial \phi}{\partial z}\,\hat{k} = (y^2z + 8xz)\,\hat{i} + (2xyz)\,\hat{j} + (xy^2 + 4x^2)\,\hat{k}$.

$\therefore \quad \left[\vec{\nabla}\phi\right]_{(-1,\ 1,\ 2)} = -14\,\hat{i} - 4\,\hat{j} + 3\,\hat{k}$.

Hence the directional derivative of ϕ at the point $(-1, 1, 2)$ in the direction of the vector $(2\,\hat{i} + \hat{j} - 2\,\hat{k})$ is

$\vec{\nabla}\phi(-1,1,2) \cdot \hat{a} = \left(-14\,\hat{i} - 4\,\hat{j} + 3\,\hat{k}\right) \cdot \left(\frac{2}{3}\,\hat{i} + \frac{1}{3}\,\hat{j} - \frac{2}{3}\,\hat{k}\right) = -\frac{28}{3} - \frac{4}{3} - 2 = -\frac{38}{3}$.

Since this is negative, ϕ is decreasing in this direction.

 # Question :- Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1, -2, -1)$ in the direction $2\,\hat{i} - \hat{j} - 2\,\hat{k}$. In what direction the directional derivative will be maximum and what is its magnitude?

 # Answer :- The unit vector in the direction of the vector $2\,\hat{i} - \hat{j} - 2\,\hat{k}$ is $\hat{a} = \frac{1}{3}\,(2\,\hat{i} - \hat{j} - 2\,\hat{k})$.

Now $\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\,\hat{i} + \frac{\partial \phi}{\partial y}\,\hat{j} + \frac{\partial \phi}{\partial z}\,\hat{k} = (2xyz + 4z^2)\,\hat{i} + (x^2z)\,\hat{j} + (x^2y + 8xz)\,\hat{k}$.

$\therefore \quad \left[\vec{\nabla}\phi\right]_{(1,-2,-1)} = 8\,\hat{i} - \hat{j} - 10\,\hat{k}$.

Hence the directional derivative of ϕ at the point $(1, -2, -1)$ in the direction of the vector $(2\,\hat{i} - \hat{j} - 2\,\hat{k})$ is

$\vec{\nabla}\phi(1,-2,-1) \cdot \hat{a} = \left(8\,\hat{i} - \hat{j} - 10\,\hat{k}\right) \cdot \left(\frac{2}{3}\,\hat{i} - \frac{1}{3}\,\hat{j} - \frac{2}{3}\,\hat{k}\right) = \frac{16}{3} + \frac{1}{3} + \frac{20}{3} = \frac{37}{3}$.

The maximum value of the directional derivative of ϕ at the point $(1, -2, -1)$ is

$\left|\vec{\nabla}\phi(1,-2,-1)\right| = |8\,\hat{i} - \hat{j} - 10\,\hat{k}| = \sqrt{64 + 1 + 100} = \sqrt{165}$.

The direction of ϕ at $(1,-2,-1)$ attained in maximum value is given by

$$\frac{\vec{\nabla}\phi(1,-2,-1)}{\left|\vec{\nabla}\phi(1,-2,-1)\right|} = \frac{1}{\sqrt{165}}\left(8\,\hat{i} - \hat{j} - 10\,\hat{k}\right).$$

 # Question :- Find the directional derivative of the function $f(x,y,z) = 2xy - z^2$ at the point $P(2,-1,1)$ in the direction towards the point $(3,1,-1)$. In what direction is the directional derivative maximum?

 # Answer :- Let $Q(3,1,-1)$ be a point. The unit vector in the direction of the vector $\overrightarrow{PQ} = \hat{i} + 2\,\hat{j} - 2\,\hat{k}$ is $\hat{a} = \frac{1}{3}\,(\hat{i} + 2\,\hat{j} - 2\,\hat{k})$.
Now $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial z}\,\hat{k} = 2y\,\hat{i} + 2x\,\hat{j} - 2z\,\hat{k}$.
$\therefore \quad \left[\vec{\nabla} f\right]_{(2,-1,1)} = -2\,\hat{i} + 4\,\hat{j} - 2\,\hat{k}$.
Hence the directional derivative of ϕ at the point $(2,-1,1)$ in the direction of $\overrightarrow{PQ}$ is $\vec{\nabla} f(2,-1,1)\cdot\hat{a} = \left(-2\,\hat{i} + 4\,\hat{j} - 2\,\hat{k}\right)\cdot\left(\frac{1}{3}\,\hat{i} + \frac{2}{3}\,\hat{j} - \frac{2}{3}\,\hat{k}\right) = -\frac{2}{3} + \frac{8}{3} + \frac{4}{3} = \frac{10}{3}$.
The maximum value of the directional derivative of f at the point $(2,-1,1)$ is $\left|\vec{\nabla} f(2,-1,1)\right| = |-2\,\hat{i} + 4\,\hat{j} - 2\,\hat{k}| = \sqrt{4+16+4} = \sqrt{24} = 2\sqrt{6}$.
The direction of f at $(2,-1,1)$ attained in maximum value is given by

$$\frac{\vec{\nabla} f(2,-1,1)}{\left|\vec{\nabla} f(2,-1,1)\right|} = \frac{1}{\sqrt{6}}\left(-\hat{i} + 2\,\hat{j} - \hat{k}\right).$$

 # Question :- Find the maximum value of the directional derivative of $\phi = xy^2 + 2yz - 3x^3z^2$ at the point $(1,-1,1)$. Find also the direction in which it occurs.

 # Answer :- Here $\phi = xy^2 + 2yz - 3x^3z^2$.
$\therefore \quad \vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\,\hat{i} + \frac{\partial \phi}{\partial y}\,\hat{j} + \frac{\partial \phi}{\partial z}\,\hat{k} = (y^2 - 9x^2z^2)\,\hat{i} + (2xy + 2z)\,\hat{j} + (2y - 6x^3z)\,\hat{k}$.
$\therefore \quad \left[\vec{\nabla}\phi\right]_{(1,-1,1)} = -8\,\hat{i} + 0\,\hat{j} - 8\,\hat{k}$.
The maximum value of the directional derivative of ϕ at the point $(1,-1,1)$ is given by $\left|\vec{\nabla}\phi(1,-1,1)\right| = |-8\,\hat{i} + 0\,\hat{j} - 8\,\hat{k}| = \sqrt{64+0+64} = 8\sqrt{2}$.
The direction of ϕ at $(1,-1,1)$ attained in maximum value is given by

$$\frac{\vec{\nabla}\phi(1,-1,1)}{\left|\vec{\nabla}\phi(1,-1,1)\right|} = \frac{1}{8\sqrt{2}}\left(-8\,\hat{i} + 0\,\hat{j} - 8\hat{k}\right) = -\frac{1}{\sqrt{2}}\left(\hat{i} + \hat{k}\right).$$

 # Question :- Find the maximum value of the directional derivative of $\phi = 2zx - y^2$ at the point $(1,3,2)$ and also the direction in which it occurs.

 # Answer :- Here $\phi = 2zx - y^2$.
$\therefore \quad \overrightarrow{\nabla}\phi = \frac{\partial\phi}{\partial x}\,\widehat{i} + \frac{\partial\phi}{\partial y}\,\widehat{j} + \frac{\partial\phi}{\partial z}\,\widehat{k} = 2z\,\widehat{i} - 2y\,\widehat{j} + 2x\,\widehat{k}.$
$\therefore \quad \left[\overrightarrow{\nabla}\phi\right]_{(1,3,2)} = 4\,\widehat{i} - 6\,\widehat{j} + 2\,\widehat{k}.$
The maximum value of the directional derivative of ϕ at the point $(1, 3, 2)$ is given by $\left|\overrightarrow{\nabla}\phi(1,3,2)\right| = |4\,\widehat{i} - 6\,\widehat{j} + 2\,\widehat{k}| = \sqrt{16+36+4} = \sqrt{56} = 2\sqrt{14}.$
The direction of ϕ at $(1, 3, 2)$ attained in maximum value is given by

$$\frac{\overrightarrow{\nabla}\phi(1,3,2)}{\left|\overrightarrow{\nabla}\phi(1,3,2)\right|} = \frac{1}{\sqrt{14}}\left(2\,\widehat{i} - 3\,\widehat{j} + \widehat{k}\right) .$$

 # Question :- Find the maximum value of the directional derivative of $\phi = x^2 + z^2 - y^2$ at the point $(1, 3, 2)$ and also the direction in which it occurs.

 # Answer :- Here $\phi = x^2 + z^2 - y^2$.
$\therefore \quad \overrightarrow{\nabla}\phi = \frac{\partial\phi}{\partial x}\,\widehat{i} + \frac{\partial\phi}{\partial y}\,\widehat{j} + \frac{\partial\phi}{\partial z}\,\widehat{k} = 2x\,\widehat{i} - 2y\,\widehat{j} + 2z\,\widehat{k}.$
$\therefore \quad \left[\overrightarrow{\nabla}\phi\right]_{(1,3,2)} = 2\,\widehat{i} - 6\,\widehat{j} + 4\,\widehat{k}.$
The maximum value of the directional derivative of ϕ at the point $(1, 3, 2)$ is given by $\left|\overrightarrow{\nabla}\phi(1,3,2)\right| = |2\,\widehat{i} - 6\,\widehat{j} + 4\,\widehat{k}| = \sqrt{4+36+16} = \sqrt{56} = 2\sqrt{14}.$
The direction of ϕ at $(1, 3, 2)$ attained in maximum value is given by

$$\frac{\overrightarrow{\nabla}\phi(1,3,2)}{\left|\overrightarrow{\nabla}\phi(1,3,2)\right|} = \frac{1}{\sqrt{14}}\left(\widehat{i} - 3\,\widehat{j} + 2\,\widehat{k}\right).$$

 # Question :- If $\overrightarrow{\nabla}\cdot\overrightarrow{E} = 0 = \overrightarrow{\nabla}\cdot\overrightarrow{H}$, $\overrightarrow{\nabla}\times\overrightarrow{E} = -\frac{\partial\overrightarrow{H}}{\partial t}$ and $\overrightarrow{\nabla}\times\overrightarrow{H} = \frac{\partial\overrightarrow{E}}{\partial t}$, then show that $\overrightarrow{\nabla}^2\overrightarrow{H} = \frac{\partial^2\overrightarrow{H}}{\partial t^2}$ and $\overrightarrow{\nabla}^2\overrightarrow{E} = \frac{\partial^2\overrightarrow{E}}{\partial t^2}$.

 # Answer :- It is given that

$$\overrightarrow{\nabla}\times\overrightarrow{E} = -\frac{\partial\overrightarrow{H}}{\partial t} , \tag{1}$$

$$\overrightarrow{\nabla}\times\overrightarrow{H} = \frac{\partial\overrightarrow{E}}{\partial t} \tag{2}$$

and
$$\overrightarrow{\nabla}\cdot\overrightarrow{E} = 0 = \overrightarrow{\nabla}\cdot\overrightarrow{H} . \tag{3}$$

Taking curl both sides of (1), we get

$$\overrightarrow{\nabla}\times\left(\overrightarrow{\nabla}\times\overrightarrow{E}\right) = \overrightarrow{\nabla}\times\left(-\frac{\partial\overrightarrow{H}}{\partial t}\right) ,$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}\cdot\vec{E}\right) - \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{E} = -\frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{H}\right),$$

$$\Rightarrow -\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{E} = -\frac{\partial}{\partial t}\left(\frac{\partial\vec{E}}{\partial t}\right), \text{ [by (3) and (2)]}$$

$$\Rightarrow \vec{\nabla}^2\vec{E} = \frac{\partial^2\vec{E}}{\partial t^2}\text{ . }(\textbf{Proved})$$

Taking curl both sides of (2), we get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right) = \vec{\nabla}\times\left(\frac{\partial\vec{E}}{\partial t}\right),$$

$$\Rightarrow \vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right) - \left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{H} = \frac{\partial}{\partial t}\left(\vec{\nabla}\times\vec{E}\right),$$

$$\Rightarrow -\left(\vec{\nabla}\cdot\vec{\nabla}\right)\vec{H} = \frac{\partial}{\partial t}\left(-\frac{\partial\vec{H}}{\partial t}\right), \text{ [by (3) and (1)]}$$

$$\Rightarrow \vec{\nabla}^2\vec{H} = \frac{\partial^2\vec{H}}{\partial t^2}\text{ . }(\textbf{Proved})$$

 # Question :- Find the angle of intersection of the spheres $x^2+y^2+z^2+4x-6y-8z-47=0$ and $x^2+y^2+z^2-29=0$, at point of intersection being $(4,-3,2)$.

 # Answer :- Let $f = x^2+y^2+z^2+4x-6y-8z-47$ and $g = x^2+y^2+z^2-29$.
Then $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial y}\,\widehat{k} = (2x+4)\,\widehat{i} + (2y-6)\,\widehat{j} + (2z-8)\,\widehat{k}$ and
$\vec{\nabla} g = \frac{\partial g}{\partial x}\,\widehat{i} + \frac{\partial g}{\partial y}\,\widehat{j} + \frac{\partial g}{\partial y}\,\widehat{k} = 2x\,\widehat{i} + 2y\,\widehat{j} + 2z\,\widehat{k}$.
$\therefore\quad \left[\vec{\nabla} f\right]_{(4,-3,2)} = 12\,\widehat{i} - 12\,\widehat{j} - 4\,\widehat{k}$ and $\left[\vec{\nabla} g\right]_{(4,-3,2)} = 8\,\widehat{i} - 6\,\widehat{j} + 4\widehat{k}$.
Let θ be the angle between the surfaces. Then θ is the angle between the normals to the surfaces. This angle at $(4,-3,2)$ is given by

$$\cos\theta = \frac{\vec{\nabla} f(4,-3,2)\cdot\vec{\nabla} g(4,-3,2)}{\left|\vec{\nabla} f(4,-3,2)\right|\;\left|\vec{\nabla} g(4,-3,2)\right|}$$

$$= \frac{\left(12\,\widehat{i} - 12\,\widehat{j} - 4\,\widehat{k}\right)\cdot\left(8\,\widehat{i} - 6\,\widehat{j} + 4\,\widehat{k}\right)}{\sqrt{144+144+16}\,\sqrt{64+36+16}} = \frac{152}{\sqrt{304}\,\sqrt{116}}\ .$$

$$\therefore\quad \theta = \cos^{-1}\left(\frac{152}{\sqrt{304}\,\sqrt{116}}\right), \text{ which is the required angle.}$$

 # Question :- Find the angle of intersection of the spheres $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$, at the point $(2, -1, 2)$.

 # Answer :- Let $f = x^2 + y^2 + z^2 - 9$ and $g = x^2 + y^2 - z - 3$.
Then $\overrightarrow{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial y}\,\widehat{k} = 2x\,\widehat{i} + 2y\,\widehat{j} + 2z\,\widehat{k}$ and
$\overrightarrow{\nabla} g = \frac{\partial g}{\partial x}\,\widehat{i} + \frac{\partial g}{\partial y}\,\widehat{j} + \frac{\partial g}{\partial y}\,\widehat{k} = 2x\,\widehat{i} + 2y\,\widehat{j} - \widehat{k}$.
$\therefore \quad \left[\overrightarrow{\nabla} f\right]_{(2,-1,2)} = 4\,\widehat{i} - 2\,\widehat{j} + 4\,\widehat{k}$ and $\left[\overrightarrow{\nabla} g\right]_{(2,-1,2)} = 4\,\widehat{i} - 2\,\widehat{j} - \widehat{k}$.
Let θ be the angle between the surfaces. Then θ is the angle between the normals to the surfaces. This angle at $(2, -1, 2)$ is given by

$$\cos\theta = \frac{\overrightarrow{\nabla} f(2,-1,2) \cdot \overrightarrow{\nabla} g(2,-1,2)}{\left|\overrightarrow{\nabla} f(2,-1,2)\right| \; \left|\overrightarrow{\nabla} g(2,-1,2)\right|}$$

$$= \frac{\left(4\,\widehat{i} - 2\,\widehat{j} + 4\,\widehat{k}\right) \cdot \left(4\,\widehat{i} - 2\,\widehat{j} - \widehat{k}\right)}{\sqrt{16+4+16}\,\sqrt{16+4+1}} = \frac{16}{6\,\sqrt{21}} = \frac{8}{3\,\sqrt{21}} \,.$$

$$\therefore \quad \theta = \cos^{-1}\left(\frac{8}{3\,\sqrt{21}}\right), \text{ which is the required angle.}$$

 # Question :- Find the unit normal vector and tangent plane to the surface $x^2yz + 4xz^2 = 6$ at the point $(1, -2, 1)$.

 # Answer :- Let $f = x^2yz + 4xz^2 - 6$.
Then $\overrightarrow{\nabla} f = \frac{\partial f}{\partial x}\,\widehat{i} + \frac{\partial f}{\partial y}\,\widehat{j} + \frac{\partial f}{\partial y}\,\widehat{k} = (2xyz + 4z^2)\,\widehat{i} + x^2z\,\widehat{j} + (x^2y + 8xz)\,\widehat{k}$.
$\therefore \quad \left[\overrightarrow{\nabla} f\right]_{(1,-2,1)} = 0\,\widehat{i} + \widehat{j} + 6\,\widehat{k}$.
The unit normal vector to the surface $f = x^2yz + 4xz^2 - 6 = 0$ at the point $(1, -2, 1)$ is

$$\widehat{n} = \frac{\overrightarrow{\nabla} f(1,-2,1)}{\left|\overrightarrow{\nabla} f(1,-2,1)\right|} = \frac{1}{\sqrt{37}}\,(\,\widehat{j} + 6\,\widehat{k}\,)\,.$$

The equation of the tangent plane at the point $\overrightarrow{r}_0 = \widehat{i} - 2\,\widehat{j} + \widehat{k}$ is

$$(\overrightarrow{r} - \overrightarrow{r}_0) \cdot \overrightarrow{\nabla} f(1,-2,1) = 0\,,$$

$$\Rightarrow \; (x-1)\,0 + (y+2)\,1 + (z-1)\,6 = 0\,,$$

$$\Rightarrow \; y + 6z = 4\,.$$

 # Question :- Find the equations of the tangent plane and normal line to the surface $xyz = 4$ at the point $\widehat{i} + 2\,\widehat{j} + 2\,\widehat{k}$.

 # Answer :- Let $f = xyz - 4$.
Then $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial y}\,\hat{k} = yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$.
$\therefore \quad \left[\vec{\nabla} f\right]_{(1,2,2)} = 4\,\hat{i} + 2\,\hat{j} + 2\,\hat{k}$.
The equation of the tangent plane at the point $\vec{r}_0 = \hat{i} + 2\,\hat{j} + 2\,\hat{k}$ is

$$(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(1,2,2) = 0\ ,$$

$$\Rightarrow\ (x-1)\ 4 + (y-2)\ 2 + (z-2)\ 2 = 0\ ,$$

$$\Rightarrow\ 2x + y + z = 6\ .$$

The equation of the normal line is

$$(\vec{r} - \vec{r}_0) \times \vec{\nabla} f(1,2,2) = \vec{0}\ ,$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y-2 & z-2 \\ 4 & 2 & 2 \end{vmatrix} = \vec{0}\ ,$$

$$\Rightarrow\ \{2(y-2)-2(z-2)\}\,\hat{i} + \{4(z-2)-2(x-1)\}\,\hat{j} + \{2(x-1)-4(y-2)\}\,\hat{k} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}\ ,$$

$$\Rightarrow\ y-2 = z-2\ \ ,\ \ 2(z-2) = x-1\ \ ,\ \ x-1 = 2(y-2)\ ,$$

$$\Rightarrow\ \frac{y-2}{1} = \frac{z-2}{1}\ \ ,\ \ \frac{z-2}{1} = \frac{x-1}{2}\ \ ,\ \ \frac{x-1}{2} = \frac{y-2}{1}\ .$$

Hence the required equation of the normal line is $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$.

 # Question :- Find the equations of the tangent plane and normal line to the surface $xz^2 + x^2y = z - 1$ at the point $(1, -3, 2)$.

 # Answer :- Let $f = xz^2 + x^2y - z + 1$.
Then $\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{i} + \frac{\partial f}{\partial y}\,\hat{j} + \frac{\partial f}{\partial y}\,\hat{k} = (z^2 + 2xy)\,\hat{i} + x^2\,\hat{j} + (2xz - 1)\,\hat{k}$.
$\therefore \quad \left[\vec{\nabla} f\right]_{(1,-3,2)} = -2\,\hat{i} + \hat{j} + 3\,\hat{k}$.
The equation of the tangent plane at the point $\vec{r}_0 = \hat{i} - 3\,\hat{j} + 2\,\hat{k}$ is

$$(\vec{r} - \vec{r}_0) \cdot \vec{\nabla} f(1,-3,2) = 0\ ,$$

$$\Rightarrow\ (x-1)(-2) + (y+3)1 + (z-2)3 = 0\ ,$$

$$\Rightarrow\ -2x + y + 3z = 1\ .$$

The equation of the normal line is

$$(\vec{r} - \vec{r}_0) \times \vec{\nabla} f(1,-3,2) = \vec{0}\ ,$$

$$\Rightarrow \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ x-1 & y+3 & z-2 \\ -2 & 1 & 3 \end{vmatrix} = \vec{0} \ ,$$

$\Rightarrow \ \{3(y+3)-(z-2)\}\,\widehat{i}+\{-2(z-2)-3(x-1)\}\,\widehat{j}+\{(x-1)+2(y+3)\}\,\widehat{k} = 0\,\widehat{i}+0\,\widehat{j}+0\,\widehat{k}\ ,$

$\Rightarrow \ 3(y+3) = z-2 \ , \ 2(z-2) = -3(x-1) \ , \ x-1 = -2(y+3) \ ,$

$$\Rightarrow \ \frac{y+3}{1} = \frac{z-2}{3} \ , \ \frac{z-2}{3} = \frac{x-1}{-2} \ , \ \frac{x-1}{-2} = \frac{y+3}{1} \ .$$

Hence the required equation of the normal line is $\frac{x-1}{-2} = \frac{y+3}{1} = \frac{z-2}{3}$.

 # Question :- If the vectors $\vec{A}$ and $\vec{B}$ be irrotational, then show that the vector $\vec{A} \times \vec{B}$ is solenoidal.

 # Answer :- A vector $\vec{a}$ is said to be solenoidal, if div $\vec{a} = 0$, i.e., $\vec{\nabla} \cdot \vec{a} = 0$.
Since $\vec{A}$ and $\vec{B}$ are irrotational, so

$$\vec{\nabla} \times \vec{A} = \vec{0} = \vec{\nabla} \times \vec{B} \ . \tag{1}$$

Again $$\vec{\nabla} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \vec{0} - \vec{A} \cdot \vec{0} = 0 \ .$$

Hence $\vec{A} \times \vec{B}$ is solenoidal vector.

 # Question :- If $\vec{a}$ and $\vec{b}$ are constant vectors and $\vec{r} = x\widehat{i} + y\widehat{j} + z\widehat{k}$, then prove that $(\vec{a} \times \vec{b}) \times \vec{r}$ is solenoidal.

 # Answer :- Let $\vec{p} = (\vec{a} \times \vec{b}) \times \vec{r} = -\vec{r} \times (\vec{a} \times \vec{b}) = (\vec{r} \cdot \vec{a})\,\vec{b} - (\vec{r} \cdot \vec{b})\,\vec{a}$.
Now $\vec{\nabla} \cdot \vec{p} = \vec{\nabla} \cdot \{(\vec{r} \cdot \vec{a})\,\vec{b} - (\vec{r} \cdot \vec{b})\,\vec{a}\} = (\vec{r} \cdot \vec{a})(\vec{\nabla} \cdot \vec{b}) - (\vec{r} \cdot \vec{b})(\vec{\nabla} \cdot \vec{a}) = 0$.
[$\because \ \vec{\nabla} \cdot \vec{a} = 0 = \vec{\nabla} \cdot \vec{b}$, as $\vec{a}$ and $\vec{b}$ are constant vectors]
Hence $\vec{p} = (\vec{a} \times \vec{b}) \times \vec{r}$ is solenoidal vector.

 # Question :- If $\vec{a}$ and $\vec{b}$ are constant vectors, prove that $\vec{\nabla} \times \left\{(\vec{r} \times \vec{a}) \times \vec{b}\right\} = \vec{b} \times \vec{a}$, where $\vec{r} = x\,\widehat{i} + y\,\widehat{j} + z\,\widehat{k}$.

 # Answer :- Here

$$(\vec{r} \times \vec{a}) \times \vec{b} = -\left\{\vec{b} \times (\vec{r} \times \vec{a})\right\}$$

$$= -\left\{\left(\vec{b} \cdot \vec{a}\right)\vec{r} - \left(\vec{b} \cdot \vec{r}\right)\vec{a}\right\} = \left(\vec{r} \cdot \vec{b}\right)\vec{a} - \left(\vec{a} \cdot \vec{b}\right)\vec{r} \ .$$

$$\therefore\ \vec{\nabla}\times\left\{(\vec{r}\times\vec{a})\times\vec{b}\right\}=\vec{\nabla}\times\left\{\left(\vec{r}\cdot\vec{b}\right)\vec{a}-\left(\vec{a}\cdot\vec{b}\right)\vec{r}\right\}$$
$$=\vec{\nabla}\times\left\{\left(\vec{r}\cdot\vec{b}\right)\vec{a}\right\}-\vec{\nabla}\times\left\{\left(\vec{a}\cdot\vec{b}\right)\vec{r}\right\}=\left(\vec{r}\cdot\vec{b}\right)\left(\vec{\nabla}\times\vec{a}\right)$$
$$+\vec{\nabla}\left(\vec{r}\cdot\vec{b}\right)\times\vec{a}-\left(\vec{a}\cdot\vec{b}\right)\left(\vec{\nabla}\times\vec{r}\right)+\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right)\times\vec{r}\ . \tag{1}$$

Since $\vec{a}$ and $\vec{b}$ are constant vectors and $\vec{r}=x\,\hat{i}+y\,\hat{j}+z\,\hat{k}$, so $\vec{\nabla}\times\vec{a}=\vec{0}$, $\vec{\nabla}\left(\vec{r}\cdot\vec{b}\right)=\vec{b}$, $\vec{\nabla}\times\vec{r}=\vec{0}$ and $\vec{\nabla}\left(\vec{a}\cdot\vec{b}\right)=\vec{0}$.
Putting these values in (1), we get

$$\vec{\nabla}\times\left\{(\vec{r}\times\vec{a})\times\vec{b}\right\}=\vec{b}\times\vec{a}\ .\ \textbf{(Proved)}$$

 # Question :- If $\vec{A}$ be a solenoidal vector, then show that $\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times\vec{\nabla}\times\vec{A}=\nabla^4\vec{A}$, i.e., curl curl curl curl $\vec{A}=\nabla^4\vec{A}$.

 # Answer :- We know that

$$\text{curl curl }\vec{A}=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{A}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{A}\right)-\vec{\nabla}^2\vec{A}\ . \tag{1}$$

Since $\vec{A}$ be a solenoidal vector, so

$$\vec{\nabla}\cdot\vec{A}=0\ . \tag{2}$$

$$\therefore\ \text{curl curl }\vec{A}=-\vec{\nabla}^2\vec{A}=\vec{H}\ ,\ \text{(say)}\ \ [\text{by (2)}]\ .$$

Again $\quad\text{curl curl }\vec{H}=\vec{\nabla}\times\left(\vec{\nabla}\times\vec{H}\right)=\vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right)-\vec{\nabla}^2\vec{H}\ .$

But $\quad\vec{\nabla}\left(\vec{\nabla}\cdot\vec{H}\right)=-\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\nabla}^2\vec{A}\right)=-\vec{\nabla}\left(\vec{\nabla}^2\vec{\nabla}\cdot\vec{A}\right)=\vec{0}$, [by (2)] .

$$\therefore\ \text{curl curl }\vec{H}=-\vec{\nabla}^2\vec{H}=-\vec{\nabla}^2\left(-\vec{\nabla}^2\vec{A}\right)=\vec{\nabla}^4\vec{A}\ .$$

Hence $\quad$ curl curl curl curl $\vec{A}=\nabla^4\vec{A}$. **(Proved)**

 # Question :- If $\vec{\nabla}\cdot\vec{D}=\rho$, $\vec{\nabla}\cdot\vec{H}=0$, $\vec{\nabla}\times\vec{D}=-\frac{1}{c}\,\frac{\partial\vec{H}}{\partial t}$ and $\vec{\nabla}\times\vec{H}=\frac{1}{c}\,\left(\frac{\partial\vec{D}}{\partial t}+\rho\,\vec{v}\right)$, then show that $\vec{\nabla}^2\vec{D}-\frac{1}{c^2}\,\frac{\partial^2\vec{D}}{\partial t^2}=\vec{\nabla}\rho+\frac{1}{c^2}\,\frac{\partial}{\partial t}(\rho\vec{v})$ and $\vec{\nabla}^2\vec{H}-\frac{1}{c^2}\,\frac{\partial^2\vec{H}}{\partial t^2}=-\frac{1}{c}\,\vec{\nabla}\times(\rho\vec{v})$, where c is a constant and t is the time variable.

 # Answer :- It is given that

$$\vec{\nabla}\cdot\vec{D}=\rho\ , \tag{1}$$

$$\vec{\nabla} \cdot \vec{H} = 0 \,, \tag{2}$$

$$\vec{\nabla} \times \vec{D} = -\frac{1}{c}\,\frac{\partial \vec{H}}{\partial t} \tag{3}$$

and
$$\vec{\nabla} \times \vec{H} = \frac{1}{c}\,\left(\frac{\partial \vec{D}}{\partial t} + \rho\,\vec{v}\right)\,. \tag{4}$$

(4) can be written as

$$\rho\,\vec{v} = c\,\left(\vec{\nabla} \times \vec{H}\right) - \frac{\partial \vec{D}}{\partial t}\,. \tag{5}$$

$$\therefore\quad \frac{\partial}{\partial t}\left(\rho\,\vec{v}\right) = c\,\frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{H}\right) - \frac{\partial^2 \vec{D}}{\partial t^2} = c\,\left(\vec{\nabla} \times \frac{\partial \vec{H}}{\partial t}\right) - \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$= -c^2\,\left\{\vec{\nabla} \times \left(\vec{\nabla} \times \vec{D}\right)\right\} - \frac{\partial^2 \vec{D}}{\partial t^2}\quad \text{[by (3)]}$$

$$= -c^2\,\left\{\vec{\nabla}\left(\vec{\nabla} \cdot \vec{D}\right) - \vec{\nabla}^2 \vec{D}\right\} - \frac{\partial^2 \vec{D}}{\partial t^2}$$

$$= -c^2\,\left\{\vec{\nabla}\rho - \vec{\nabla}^2 \vec{D}\right\} - \frac{\partial^2 \vec{D}}{\partial t^2}\quad \text{[by (1)]}$$

$$\text{or,}\quad \vec{\nabla}^2 \vec{D} - \frac{1}{c^2}\,\frac{\partial^2 \vec{D}}{\partial t^2} = \vec{\nabla}\rho + \frac{1}{c^2}\,\frac{\partial}{\partial t}\left(\rho\,\vec{v}\right).\ \textbf{(Proved)}$$

Taking cross product both sides of (5), we get

$$\vec{\nabla} \times \left(\rho\,\vec{v}\right) = c\,\vec{\nabla} \times \left(\vec{\nabla} \times \vec{H}\right) - \frac{\partial}{\partial t}\,\left(\vec{\nabla} \times \vec{D}\right)$$

$$= c\,\left\{\vec{\nabla}\left(\vec{\nabla} \cdot \vec{H}\right) - \vec{\nabla}^2 \vec{H}\right\} + \frac{1}{c}\,\frac{\partial^2 \vec{H}}{\partial t^2}\ \text{[by (3)]}$$

$$= -c\,\vec{\nabla}^2 \vec{H} + \frac{1}{c}\,\frac{\partial^2 \vec{H}}{\partial t^2}\ \text{[by (2)]}$$

$$\text{or,}\quad \vec{\nabla}^2 \vec{H} - \frac{1}{c^2}\,\frac{\partial^2 \vec{H}}{\partial t^2} = -\frac{1}{c}\,\vec{\nabla} \times \left(\rho\vec{v}\right).\ \textbf{(Proved)}$$

EXERCISE

1. If $\phi = x^3 + y^3 + z^3$, find the directional derivative of ϕ at the point $(1, -1, 2)$ in the direction of the vector $\hat{j} + \hat{k}$.
2. Prove that $\vec{\nabla} \cdot (r^3 \vec{r}) = 6r^3$, where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$.
3. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\vec{c}$ is a constant vector, prove that $\text{curl}\{(\vec{c} \cdot \vec{r})\,\vec{c}\} = \vec{0}$.
4. If f and g are both differentiable scalar functions of x, y and z, then prove that $\vec{\nabla} \cdot \left(f\,\vec{\nabla} g - g\,\vec{\nabla} f\right) = f\,\nabla^2 g - g\,\nabla^2 f$.
5. In what direction from the point $(1, 3, 2)$ is the directional derivative of $f = 2xz - y^2$ a maximum? What is the magnitude of this maximum?
6. Find the directional derivative of the function $\psi = x^2y^3 + y^2z^3 + z^2x^3$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at the point $(1, 1, 1)$.
7. If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, $r = |\vec{r}|$ and $\psi(x, y, z) = (x^2 + y^2 + z^2)\ e^{-\sqrt{x^2+y^2+z^2}}$, prove that $\vec{\nabla}\psi = (2 - r)e^{-r}\vec{r}$ and hence show that $\vec{\nabla}^2\psi = (r^2 - 6r + 6)e^{-r}$.
8. If $\vec{V} = \sin y\ \hat{i} + \sin x\ \hat{j} + e^z\ \hat{k}$, then show that $\vec{V}$ is neither solenoidal nor irrotational.
9. If $\vec{\nabla} \times \vec{f} = \vec{0}$, then prove that $\alpha = 0$ or $\beta = -1$, where $\vec{f} = (xyz)^\alpha (x^\beta\ \hat{i} + y^\beta\ \hat{j} + z^\beta\ \hat{k})$.
10. Find the constants a, b and c such that the vector $\vec{A} = (x + ay + 4z)\ \hat{i} + (2x - 3y + bz)\ \hat{j} + (cx - y + 2z)\ \hat{k}$ is irrotational.
11. Show that $\vec{A} = 2xyz^3\ \hat{i} + x^2z^3\ \hat{j} + 3x^2yz^2\ \hat{k}$ is irrotational. Find the scalar potential ϕ such that $\vec{A} = \vec{\nabla}\phi$.
12. If ϕ and ψ are differentiable scalar fields, then prove that $\vec{\nabla}\phi \times \vec{\nabla}\psi$ is solenoidal.

ANSWERS

1. $\frac{15}{2}\sqrt{2}$ **5.** $4\hat{i} - 6\hat{j} + 2\hat{k},\ \sqrt{56}$ **6.** $\frac{15}{7}\sqrt{14}$ **10.** $a = 2,\ b = -1,\ c = 4$
11. $\phi = x^2yz^3$