Superconductivity

P. Mandal

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Syllabus: Experimental Results. Critical Temperature. Critical magnetic field. Meissner effect. Type I and type II Superconductors, London's Equation and Penetration Depth. Isotope effect.

This is a lecture note and it does not include the details of the topics. For an insight, the students are advised to consult text books and other references as recommended in the class. While preparing this note, I have exclusively taken the help from many sources. In particular to mention, are *Solid State Physics* by R. K. Puri and V. K. Babbar, *Introduction to Solid State Physics* by Ashcroft and Mermin, e-PGPathshala by MHRD, Govt. of India and other web resources. I have frequently quoted the statements and explanations which I could not make lucid as they appear in these references.

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1 Introduction

The phenomenon of superconductivity was discovered in 1911 by Kamerlingh Onnes[1] and it has emerged as one of the most exciting fields of technological applications over past few decades. While measuring the resistivity of solid mercury at the cryogenic temperature using the recently-discovered (by himself) liquid helium as refrigerant, Onnes observed the resistivity abruptly disappeared at temperature of 4.2 K [fig. 1]. Onnes reported the observation as "Mercury has passed into a new state, which on account of its extraordinary electrical properties may be called the superconductive state". Other than mercury, several metallic elements like Al, Cd, Ga exhibit superconductivity near or below 2 K. Semiconductors like Si, Ge, under suitable conditions can become superconductors. Apart from the elements there are thousands of alloys which exhibit superconductivity even at temperature as high as 150 K. The transition from the normal state to the superconducting state is a sharp one in bulk specimens of the matters and in absence of applied magnetic field

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[1] Kamerlingh Onnes won the Nobel prize in physics in 1913 for "his investigations on the properties of matter at low temperatures which led, inter alia, to the production of liquid helium".

occurs at a particular temperature which is known as the **critical temperature** (T_c). Such a transition is regarded as second order phase transition. T_c is the characteristic of the material and it ranges from sub-Kelvin to hundred Kelvin. In presence of magnetic field, the transition occurs at a temperature lower than T_c and the nature of transition changes from second order to first order.

Few properties of the material remain unchanged as the transition from the normal to the superconducting state takes place. For examples,

- Elastic properties
- Thermal expansion behaviour
- Photoelectric properties
- The internal arrangement of crystal lattice, as confirmed by X-ray diffraction pattern before and after such a transition

The following properties are affected due to the transition into the superconducting state.

- Magnetic properties
- Electrical properties – electrical resistivity tends to zero at $T = T_c$.
- All thermo-electric effects disappear for $T \leq T_c$.
- Specific heat shows a discontinuous change
- Entropy shows a decrease for $T \geq T_c$.

Superconducting state is a distinct phase of matter having characteristic electrical, magnetic and thermodynamic properties are discussed in the following sections.

2 Electrical Properties

At low temperature the resistivity of a normal metal containing non-magnetic impurities varies as $\rho(T) = \rho_0 + BT^5$, where ρ_0 and B are constant. Thus, as the temperature approaches the absolute zero, the resistivity of normal metal approaches the constant value ρ_0 which is basically determined by the impurity. However, a superconductor behaves as if it had no measurable dc electrical resistivity [fig. 2] the critical temperature T_c. Current established in superconductor shows no discernible decay in absence of any driving field. Well below the critical temperature, superconductors also respond to an ac electric field

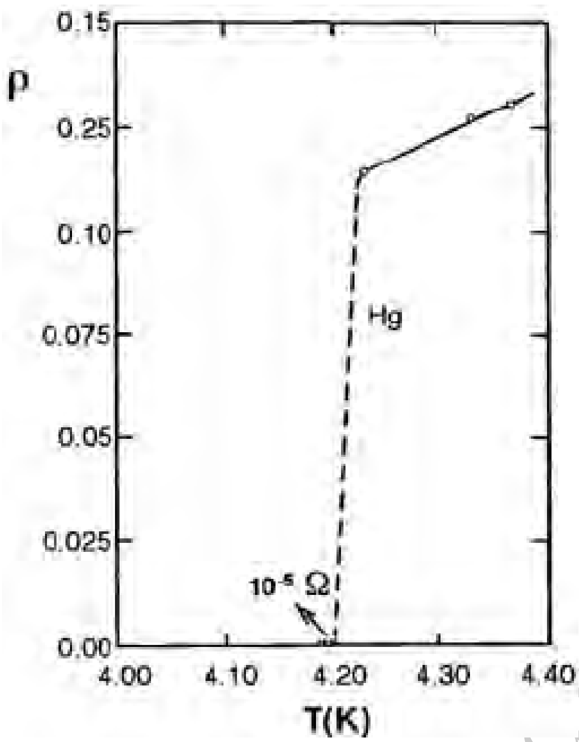

Figure 1: Disappearance of the electrical resistivity (ρ) at the liquid helium temperature as observed by Kamerlingh Onnes.

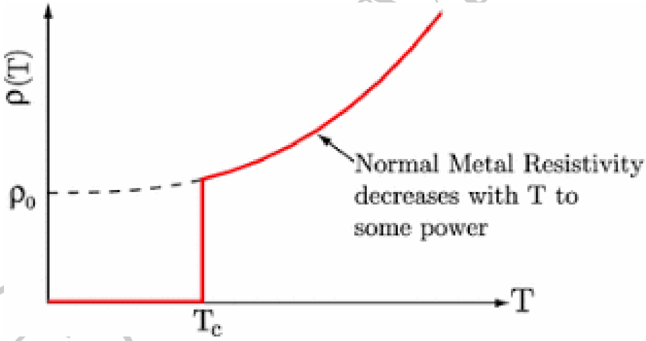

Figure 2: Variation of Low temperature resistivity of normal metal and superconductor (in zero magnetic field) with temperature.

without dissipation provided that the frequency is not too large. A ‘*supercurrent*’ can flow across an insulating junction which is known as the **Josephson Effect**. The phenomenon of superconductivity is destroyed by application of a sufficiently large magnetic field, the **critical magnetic** field (H_c). This phenomenon is known as **Meissner effect** and is discussed in detail in the following section. The superconducting state is also destroyed as the current exceeds a **critical current** (I_c), determined by the nature and geometry of the specimen[2]. This is known as **Silsbee efect**.

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[2] critical current in a 1-mm wire is as large as 100 amp.

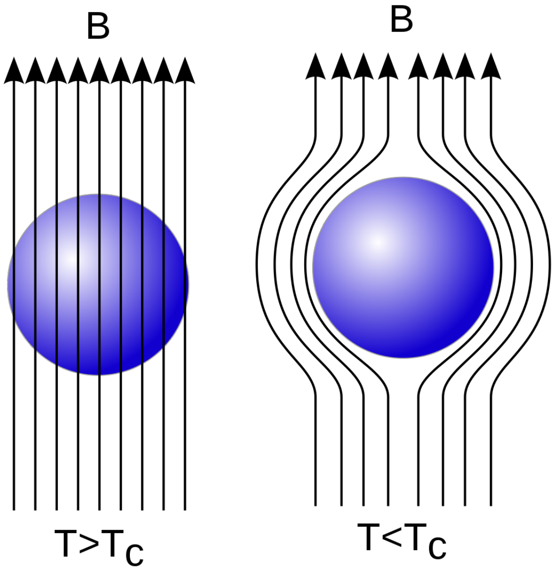

Figure 3: Meissner effect: The magnetic flux is expelled at $T < T_c$ depicting perfect diamagnetism. (Source: wikipedia)

3 Magnetic Properties: Meissner Effect

Below the critical temperature T_c and below the critical magnetic field H_c, superconducting materials expel magnetic flux through it [fig. 3] i.e., the materials behave as perfect diamagnet ($M = -H$, susceptibility $\chi = -1$, relative permeability $\mu_r = 0$). This phenomenon is known as the **Meissner Effect**[3]. This 'diamagnetic' property is more fundamental than the zero resistance property for the superconducting materials. Such a strong diamagnetic property can cause the material to levitate in space - a phenomenon known as Superconducting Magnetic Levitation. The critical magnetic field H_c for superconducting materials is dependent on the temperature of the superconducting state ($T < T_c$) by the following equation:

$$H_c(T) = H_0 \left[1 - \left(\frac{T}{T_c}\right)^2\right] \tag{1}$$

where H_0 is the critical magnetic field at the absolute zero temperature ($T = 0$ K). As the external field strength H is increased beyond the critical value $H_c(T)$, the superconductivity breaks down and the specimen reverts back to the normal state. Superconductors are divided into two types according to how this breakdown occurs: type I and type II superconductors.

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[3] Walther Meissner and Robert Ochsenfeld discovered this phenomenon in 1933 in superconducting tin and lead samples, and also known as the Meissner–Ochsenfeld effect. Phenomenological explanation of Meissner effect came in 1935 due to Fritz and Heinz London.

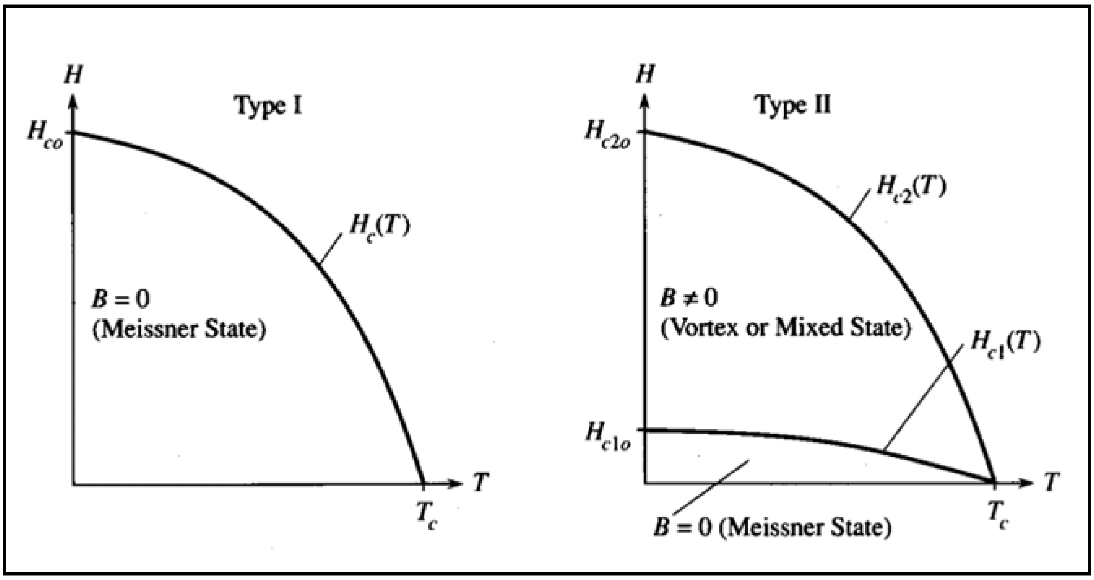

Figure 4: The phase boundary between the superconducting and normal states in type I and type II specimens. The boundary is defined by the curve $H_c(T)$ [eq. 1].

Why is the strong diamagnetic property more fundamental than the zero resistance property of superconducting materials?

Perfect conductors (conductivity $\sigma = \infty$) show the zero resistance property like the superconductors. However, unlike the superconductors perfect conductors do not exhibit the Meissner effect. The electric field inside a perfect conductor $\vec{E} = \vec{j}/\sigma \to 0$ and by Faraday's law ($\vec{\nabla} \times \vec{E} = -\partial\vec{B}/\partial t$) is satisfied by any time-independent magnetic field $\vec{B}$. Thus perfect conductivity implies a time-independent magnetic field in the interior but not the fact that the magnetic field is zero. However, the magnetic field is zero in the interior of a bulk superconducting material.

3.1 Type I and Type II Superconductors

Type I superconductors expel all magnetic flux for $H < H_c(T)$ and $T < T_c$. Superconductivity is abruptly destroyed at $H > H_c(T)$, the entire specimen reverts back to the normal state and the magnetic field lines penetrate perfectly in the interior. Transition from superconducting to the normal state for $H > H_c(T)$ is basically a first order phase transition and the phase diagram is described in figure 4 in H-T plane. The field penetration is often described in reference to diamagnetic magnetization (M) versus the applied field (H) [fig. 5]. Pure metals like Al, Pb, Hg exhibit such features and are examples of type I superconductors.

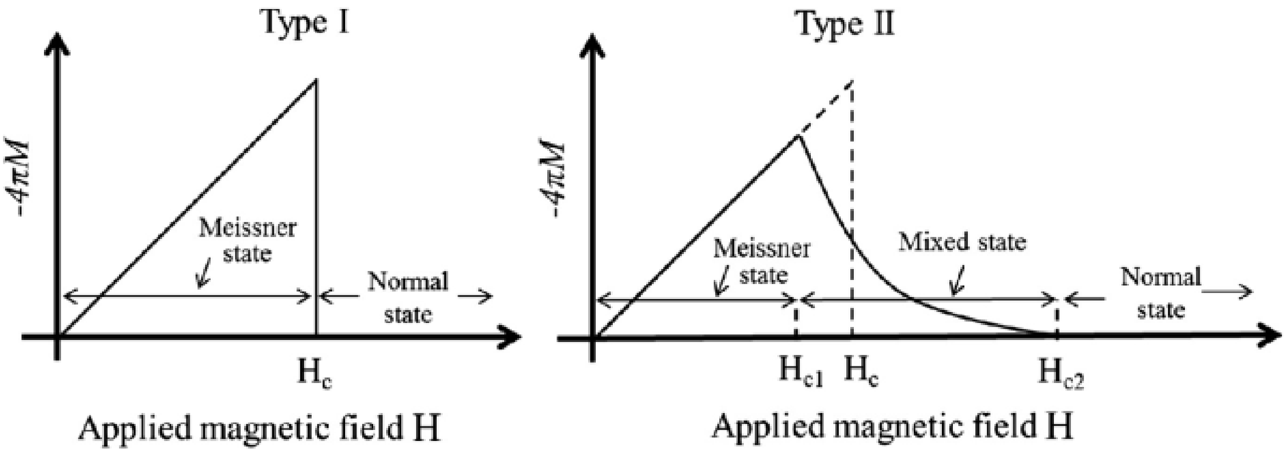

Figure 5: Diamagnetic magnetization ($-M$) versus the applied field (H) for type I and type II superconductors. Source: hyperphysics

For type II superconductors there exists two critical fields H_{c1} and H_{c2}. These materials expel all magnetic flux for $H < H_{c1}$ at temperature $T < T_c$. The magnetic flux is partially expelled for $H > H_{c1}$, and the materials enter a state of mixed normal and superconductivity which is known as the vortex region. Superconductivity disappears at $H > H_{c2}$. Alloys like NbTi, YBCO exhibit such features and are examples of type II superconductors.

3.2 The London Equation

In Meissner effect it is discussed that the metals in the superconducting state do not permit any magnetic field in its interior below the critical field H_c. In 1935, F. London and H. London first examined the fundamental fact in a quantitative way. The basic assumption of the approach is that only a fraction n_s/n of the total conduction electron density participates in the supercurrent at a temperature $T < T_c$. n_s is the density of superconducting electrons which is a function of the temperature. As the temperature is reduced below T_c, n_s increases while at temperature beyond T_c, n_s vanishes. The supercurrent flows due to the superconducting electrons with no resistance whatever, it will carry the entire current induced by any small transitory electric field and normal electrons (of density $n - n_s$) will remain quite inert. It is therefore, the normal current is not taken into account in the following.

If an electric field arises momentarily within the superconductor, the superconducting electrons will be accelerated freely and their average velocity will satisfy the following equation:

$$m\frac{d\vec{v}_s}{dt} = -e\vec{E} \tag{2}$$

The current density due to these superconducting electrons is $\vec{j}_s = -en_s\vec{v}_s$ and hence,

$$\frac{d\vec{j}_s}{dt} = -n_s e \frac{d\vec{v}_s}{dt} = \frac{n_s e^2}{m}\vec{E} \tag{3}$$

From equation 3,

$$\begin{aligned}\vec{\nabla} \times \frac{d\vec{j}_s}{dt} = \frac{n_s e^2}{m}\vec{\nabla} \times \vec{E} &= -\frac{n_s e^2}{m}\frac{\partial \vec{B}}{\partial t} \\ \Rightarrow \frac{\partial}{\partial t}\left(\vec{\nabla} \times \vec{j}_s + \frac{n_s e^2}{m}\vec{B}\right) &= 0 \qquad (4)\end{aligned}$$

where we have substituted $\vec{\nabla} \times \vec{E}$ by $-\frac{\partial \vec{B}}{\partial t}$ by using Faraday's law. Let us recall the Maxwell equation

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}_s \tag{5}$$

Note, by equation 5, a static magnetic field $\vec{B}$ determines a static current density $\vec{j}_s$; and by equation 4, any time-independent $\vec{B}$ and $\vec{j}_s$ are trivial solutions. Thus equations 4 and 5 are consistent with an arbitrary magnetic field which can exist within the perfect conductor. This is, however, incompatible with the Meissner effect that permits no magnetic field in the interior of superconductor. F. London and H. London found that Meissner effect could be explained by restricting the full set of solutions of equation 4 to those which obey

$$\vec{\nabla} \times \vec{j}_s = -\frac{n_s e^2}{m}\vec{B} \tag{6}$$

Equation 6 is known as **London equation**.

From equation 5,

$$\begin{aligned}\vec{\nabla} \times \left(\vec{\nabla} \times \vec{B}\right) &= \mu_0 \vec{\nabla} \times \vec{j}_s \\ \Rightarrow \vec{\nabla}\left(\vec{\nabla} \cdot \vec{B}\right) - \nabla^2 \vec{B} &= -\frac{\mu_0 n_s e^2}{m}\vec{B} \\ \Rightarrow \nabla^2 \vec{B} &= \frac{\vec{B}}{\lambda^2} \qquad (7)\end{aligned}$$

where we have plugged in London equation (equation 6) into equation 7 and used the Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$. Here $\lambda = \left(\frac{m}{\mu_0 n_s e^2}\right)^{1/2}$ is known as *London penetration depth* which describes that magnetic field in superconductors can exists only within a layer of thickness of the surface (of the order of $10^2 - 10^3$ Å). Near the critical temperature T_c, n_s approaches zero and λ increases implying that Meissner effect vanishes.[4]

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[4] For detailed study, readers are referred to *Solid State Physics* by Ashcroft and Mermin.

Problem 1: Show that the magnetic field penetration decays exponentially in a semiinfinite superconductor occupying the half space $x > 0$.

Solution: In case of a semiinfinite superconductor occupying the half space $x > 0$, equation 7 can be written as

$$\frac{d^2B}{dx^2} = \frac{B}{\lambda^2} \tag{8}$$

and the general solution is $B(x) = A\exp(x/\lambda) + B\exp(-x/\lambda)$. As the solution holds for $x > 0$, $A = 0$. Assuming $B(x) = B(0)$ on the surface ($x = 0$) of the superconductor, the field penetration is described by $B(x) = B(0)\exp(-x/\lambda)$ which shows that the field decays exponentially. The field decays to $1/e$ times its value at the surface at a depth $x = \lambda$.

Problem 2: Given the London penetration depth in a superconducting specimen is 500 Å. Estimate the superconducting electron density (n_s) and compare it with the free electron density (n) in a metal.

4 Isotope Effect

The critical temperature (T_c) depends on the isotopic mass (M) as $T_c \propto M^{-1/2}$. Thus, larger the isotopic mass, lower is the critical transition temperature. The mean squared amplitude of lattice vibration at low temperature is also proportional to $M^{-1/2}$ and so is the Debye temperature (θ_D). Thus T_c/θ_D is constant. Actually, $T_c \propto M^{-\beta}$, where $\beta = 0.5$ or some different value.

Problem 3: The transition temperature of ordinary mercury of atomic mass 200.59 is 4.153 K, find out the transition temperature of mercury having atomic mass 204.

Solution: The transition temperature $T_{c1} = 4.153$ K for ordinary mercury of atomic mass $M_1 = 200.59$ u. Let the transition temperature of mercury having atomic mass $M_2 = 204$ u is T_{c2}.

$$\begin{aligned} \therefore \frac{T_{c2}}{T_{c1}} &= \sqrt{\frac{M_2}{M_1}} \\ \Rightarrow T_{c2} &= T_{c1}\sqrt{\frac{M_2}{M_1}} = 4.153 \times \sqrt{\frac{200.59}{204}}\,\text{K} = 4.118\,\text{K} \end{aligned}$$

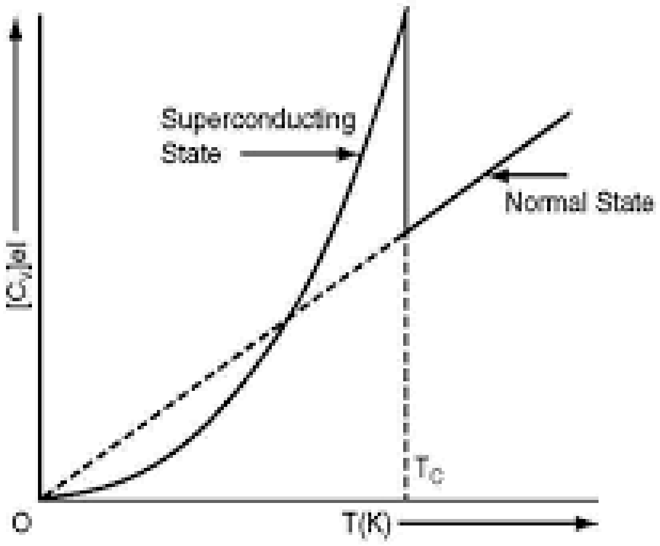


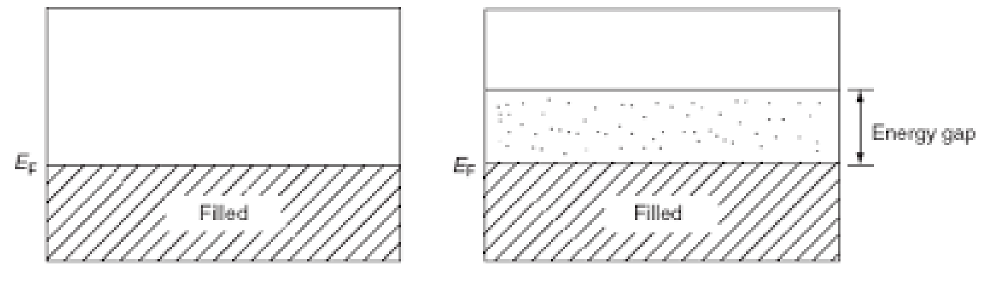
Figure 6: Variation of specific heat with temperature in the normal and superconductive states.

5 Thermal Properties

Specific heat for normal metal at low temperature varies with the temperature following the relation $C_V = aT + bT^3$, where a and b are constants. The first term accounts for the specific heat of electrons in the metal and the second term accounts for the lattice vibrations. For superconductors C_V shows a jump at the transition temperature T_c and it falls almost exponentially as the temperature is decreased below T_c in the superconducting state. This exponential behaviour is an indication of the existence of a finite gap E_g in the energy spectrum of electrons separating the ground state from the lowest state.

$$C_{sc}(T) = A\exp(-E_g/2k_BT) \tag{9}$$

The energy gap of superconductors is of entirely different nature than the energy gap in insulators. In superconductors, the energy gap is due to electron-electron interaction in Fermi gas whereas in insulators or semiconductors the energy gap is caused by electron lattice interaction. In insulators, the gap prevents the flow of electrical current. Energy must be added to lift the electrons from the valence band to the conduction band before the current can flow. In a superconductor, on the other hand, the current flows despite the presence of energy gap. In a superconductor the electrons in the excited state above the gap behave as normal electrons. The electrons in a superconducting material behave as normal electrons above temperature T_c, but the electrons are paired (Cooper pair) at temperature below T_c with total energy less than E_F. The difference between the normal state and the paired state electrons appears as the energy gap E_g at the Fermi surface. The expression for the energy gap is



(a) Normal (b) Superconducting

Figure 7: Energy gap in normal and superconductive states.

as in the following.

$$E_g = 2bk_BT_c,\ 2b \simeq 3.5 \tag{10}$$

For aluminium, $T_c = 1.2$ K. Therefore, the energy band gap for aluminium is $E_g = 3.5 \times 1.38 \times 10^{-23} \times 1.2$ J $= 0.362$ meV.

6 BCS Theory: Outlines

Microscopic theory on superconductivity was put forwarded by John Bardeen, Lee Cooper and Bob Schrieffer, and known as the BCS theory. Fundamental postulate of BCS theory is that when an attractive interaction between two electrons by means of phonon exchange dominates the repulsive coulomb interaction then the superconducting state is formed. During an interaction of an electron with a positive ion of the lattice through electrostatic coulomb force, some electron momentum get transferred. As a result, these ions set up elastic wave in the lattice due to distortion. If another electron happens to pass through this region then the interaction between two occurs which in its effect lowers the energy of the second electron. The two electrons interact via the lattice distortion or the phonon field resulting in the lowering of energy of the electron which implies the force between two electrons is attractive. This interaction is strongest when two electrons have equal and opposite moments and spin and this pair is known as **cooper pair**. When the temperature of the specimen is lowered, if the attractive force between two electrons via a phonon exceeds the Coulomb repulsion between them, then a weakly bound cooper pair is formed having the binding energy of the order of 1 meV. The energy of the Cooper pair is less than the energy of the pair in free state. The binding energy of the cooper pair is called the energy bang gap (E_g). When $h\nu > E_g$ strong absorption occurs as the cooper pairs break apart. The electrons in cooper pair have opposite spins so the total spin of the pair is zero. As a result cooper pairs are bosons whereas electrons are fermions.

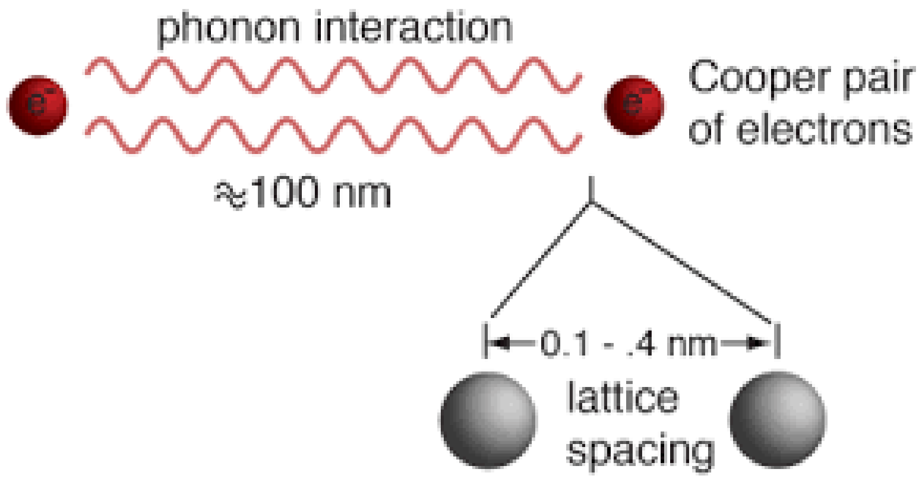


Figure 8: Cooper pair formation via phonon interaction. Source: hyperphysics

7 High T_c Superconductors

HTSC–a new class of superconducting materials with high T_c values. Bear extraordinary superconducting and magnetic properties and have great potential for wide-ranging technological applications. Ordinary metal superconductors have T_c below 20 K where as HTSC have been observed with T_c as high as 138 K. HTSC materials are not metal or intermetallic compounds but oxides of copper in combination with other elements. In 1983, 1987 and 1988 materials with T_c upto 40 K, 93 K, 125 K discovered respectively.

8 Applications

- High-field magnet applications.
- NMR – Nuclear Magnetic Resonance (medical diagnostics and spectroscopy).
- Magnetic levitation – high speed train.
- Magnetic shielding.
- High and stable magnetic field production – mass spectrometer (Penning trap), colliders etc.
- Energy storage and electric power transmission.
- SQUIDS – Superconducting Quantum Interference Devices.
- Josephson devices – square-law detector, parametric amplifier, mixer.
- Semiconductor – superconductor hybrids (A-D converter) for computer application.

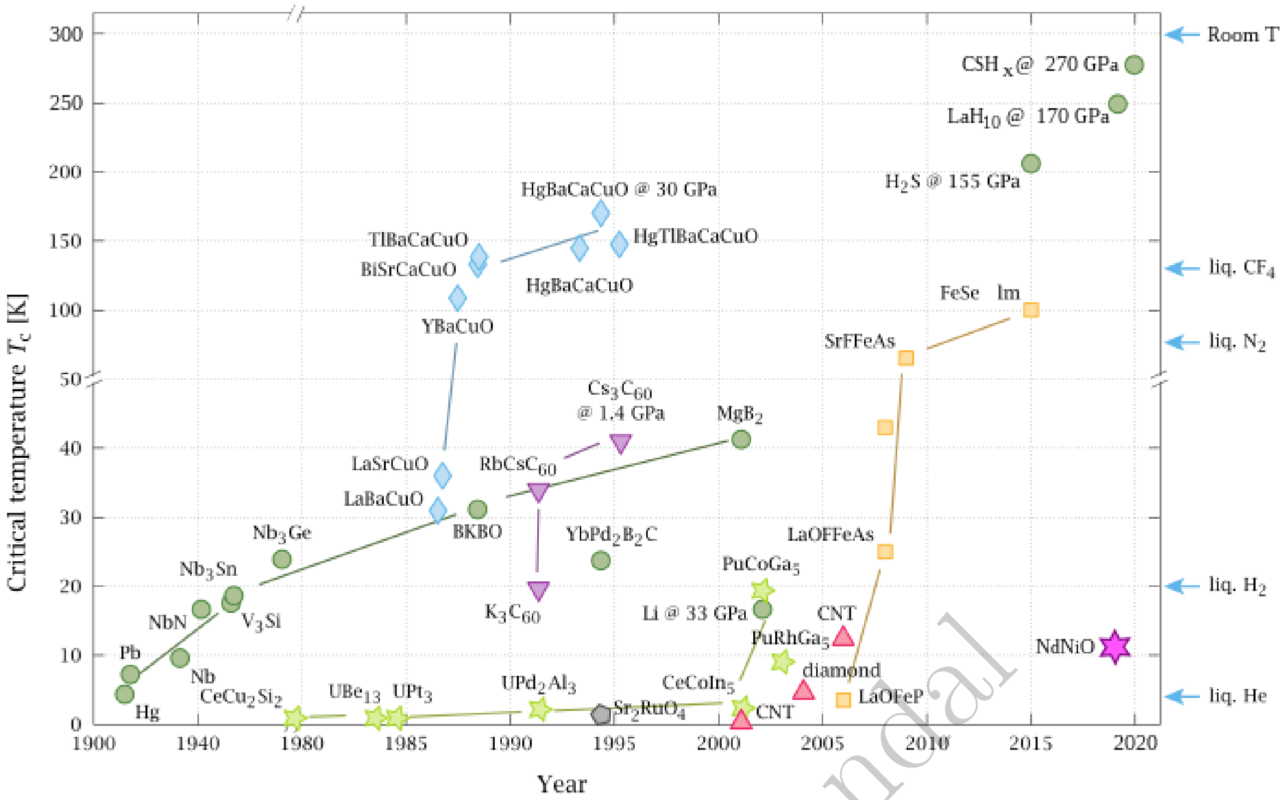


Figure 9: Timeline of superconductor discoveries. On the right one can see the liquid nitrogen temperature, which usually divides superconductors at high from superconductors at low temperatures. Source: wikipedia

- Superconducting quantum computing.
- Optoelectronics applications.

9 Review of CU Exam. Papers

CU-2022 (CBCS)

1. Perfect diamagnetism and zero resistivity of a superconductor are the two mutually exclusive properties. - Explain. [2]
2. Distinguish between type I and type II superconductors with the help of M–H plot. [2]

CU-2021 (CBCS)

1. What does the existence of energy gap in a superconductor imply? [2]
2. What is the relation between isotopic mass and transition temperature in a superconductor? Show the variation of energy gap with temperature. [2+2]
3. Write down the expression for penetration of external magnetic field inside a superconductor. [1]

4. In an experiment, a niobium (Nb) wire of radius 0.25 mm is immersed in liquid helium ($T = 4.2$ K) and required to carry a current of 300 A. It is given that $H_C(0) = 0.20$ T and the critical transition temperature T_C of Nb is 9.3 K. Will the wire remain superconducting? [3]

   [**Hint.** The critical current (I_c) and the critical magnetic field (H_c) in a superconducting wire of radius R are related as $H_c = 2\pi R I_c$. This follows from Ampere's law and known as Silsbee effect.]

CU-2020 (CBCS)

1. Sketch the temperature variation of specific heat of a superconductor and a normal metal in the same graph. [2]
2. What is Meissner Effect? The perfect diamagnetism and Zero resistivity of a superconductor are the two mutually exclusive properties—Explain. Discuss the difference between type-I and type-II superconductors. [1+2+2]
3. Estimate the London penetration depth for tin (density 7300 kg m^{-3}). Given the atomic weight = 118.7 u, $T_C = 37$ K and effective mass of electron $m^* = 19m_e$. [3]
4. Briefly explain how BCS theory accounts for the superconducting state. [2]

CU-2021

1. What is isotope effect in superconductivity? [2]
2. From which experiment, one can get the idea of the energy gap in superconductivity? How does the energy gap depend on temperature? [2]
3. What are Cooper pairs? What can you say about its spin state? [2]
4. Write down two important differences between normal superconductors and high T_c superconductors. [2]

CU-2020

1. Is Meissner effect consistent with the disappearance of resistivity in a superconductor? Explain. [2]
2. What are the differences between semiconductor energy gap and the superconducting energy gap? [2]
3. Explain briefly the Meissner effect with suitable diagram. Show that the magnetic field decays inside the superconductor exponentially with a characteristic length scale. [1+3]

4. What is the physical implication of isotope effect in superconductivity? The critical temperature for mercury with isotope mass 202 is 4.159 K. Determine its critical temperature when its isotope mass is 200.7. [2]

CU-2019

1. Write down two important differences between normal superconductor and High-T_c superconductor. [2]

2. Sketch the specific heat of a superconductor and a normal metal as a function of temperature in the same graph. What information can be obtained from the above graph? [2+1]

3. Distinguish between type I and type II superconductors with the help of $M - H$ plot. [2]

4. A given superconductor has critical magnetic fields 1.4×10^5 A/m and 4.2×10^5 A/m at 14 K and 13 K respectively. Compute the transition temperature and the critical magnetic field at 0 K. [3]

CU-2018

1. What are the differences of energy gap seen in superconductor and semiconductor? [2]

2. Explain 'isotope effect' in superconductivity. Briefly discuss its significance. [2+1]

3. Derive the behavior of magnetic field inside the superconductor. Hence, define the characteristic length scale. [2+1]

CU-2017

1. Sketch the specific heat of a superconductor and normal metal as a function of temperature. (Indicate the critical transition temperature in the graph). [2]

2. Explain briefly the Meissner effect with a suitable diagram. [2+1]

3. Calculate the wavelength of the photon which will be required to destroy the superconductivity in Aluminium having critical transition temperature 1.2 K. In which region of electromagnetic spectrum does it belong? [1+1]

CU-2016

1. Is Meissner effect consistent with disappearance of resistivity in a superconductor? Explain. [2]

2. In which ways, the energy gaps seen in superconductor are different from semiconductor? How does the energy gap vary with temperature? Sketch the variation of specific heat of a superconductor with temperature (show also the critical transition temperature T_c in the graph). [2+1+2]

CU-2015

1. Sketch the specific heat of a superconductor and normal metal as a function of temperature in the same graph. [2]
2. Distinguish between type I and type II superconductors with the help of M–H plot. [2]
3. What is the implication of isotope effect? Given that the transition temperature of ordinary mercury (Hg) of atomic mass 200.59 is 4.153 K, find out the transition temperature of mercury having atomic mass 204. [2+1]

CU-2014

1. What is the experimental observation of the presence of an energy gap in a superconductor? [2]
2. Explain briefly the Meissner effect with a suitable diagram. Show that the magnetic field decays inside the superconductor exponentially with a characteristic length scale. [(2+1)+3]

CU-2013

1. For a superconductor, what do you mean by critical temperature and critical field? [2]
2. Explain the meaning of energy gap in a superconductor. [2]
3. What is isotope effect? What is its importance in the theory of superconductivity? [2+1]