Lecture Notes on Group Theory

Normal subgroups and Group homomorphisms

B.Sc Honours Course(MTMA Module X)

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Contents

1 Introduction

Recall that a group $(G, \cdot)$ is a non-empty set together with a binary operation defined on G

(i)satisfying the associative law, i.e., $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in S$,

(ii) having an identity e, satifying $a \cdot e = a = e \cdot a \ \forall a \in G$

(iii) each element $a \in G$ has an inverse a' satisfying $a \cdot a' = a' \cdot a = e$.

A gorup G is said to be abelian if $a \cdot b = b \cdot a \ \forall a, b \in G$.

Some examples of abelian group are $(\mathbb{R}, +)$, $(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{C}, +)$, $(K_4, *)$ the Klein's 4- group.

On the other hand for any $n \geq 3$ the symmetric group S_n is a non abelian group.

For any $n \in \mathbb{Z}$ and for any $a \in G$ we can define a^n by

$$a^n = \begin{cases} a \cdot a \cdot \cdot a \ (n \text{ times if } n > 0) \\ \text{e}(if\text{n} = 0) \\ \text{a}^{-1} \cdot a^{-1} \cdot \cdot a^{-1} \text{ ,if } n < 0 \end{cases}$$

A group G is said to be a cyclic group generated by an element a and we write $G =< a >$ if $G = \{a^n : n \in \mathbb{Z}\}$. For a finite group G of order n G is cyclic iff G has an element a (which will be a generator) of order n. The group Z_n of classes congruent modulo n is a cyclic group under usual addition. The group $(\mathbb{Z}, +)$ is an infinite cyclic group with 1 and -1 are two generators.

A nonempty subset H of a group G is called a subgroup of G if H is itself a group under the restriction of the binary operation $\cdot$.

Then we have already proved that a nonempty subset H of a group G is a subgroup of G iff $a \cdot b^{-1} \in H \forall a, b \in H$. Any subgroup of an abelian group is abelian and any subgroup of a cyclic group is cyclic. The intersection of two subgroups of a group is also a subgroup, but the union of two subgroups will be a subgroup iff one subgroup is contained in the other. In Abstract Algebra II, we have define a left coset and right coset of a subgroup. Given a subgroup H of a group $(G, \cdot)$ two left cosets aH and bH are equal iff $a^{-1} \cdot b \in H$. The set of all left cosets (also right cosets) of H forms a partition of G. For a finite group G we have proved Lagrange's theorem: order of any subgroup of a finite group divides the order of the group. Indeed, for a subgroup H of a finite group G, the relation $|G| = |H| \times [G : H]$ holds, where $[G : H]$ is the number of distinct left(or right) cosets of H, called the index of H in G.

We shall now introduce a special type of subgroups H of a group G to construct a new group, to be called quotient group.

2 Normal Subgroups

Definition 2.1. *A subgroup H of a group $(G, \cdot)$ is said to be a normal subgroup of G if any left coset of H is equal with the corresponding right coset of H i.e $aH = Ha$ $\forall a \in G$.*

Example 2.2. *(i) For any group G the trivial subgroups $\{e\}$ and G are always normal subgroups.*
(ii) Every subgroup of an abelian group is a normal subgroup. Indeed, if H is a subgroup of an abelian group G, then for any $a \in G$ the set $\{a \cdot h : h \in H\} = \{h \cdot a : h \in H\}$.
(iii) Consider the quaternion group Q_8 generated by two elements a, b where $o(a) = 4$, $a^2 = b^2$ and $b \cdot a = a^3 \cdot b$. Then it can be shown that Q_8 is a nonabelian group of order 8 whose every subgroup is normal.
(iv) Consider the symmetric group S_3 which is a nonabelian group of order 6 containing the permutations over$\{1, 2, 3\}$. Here the subgroup $H_1 = \{e, (1, 2)\}$ is not a normal subgroup. In fact it can be easily verified that $(1, 3)H_1 \neq H_1(1, 3)$.
(v) For any group G the center $Z(G)$ is a normal subgroup.

Example 2.3. *Suppose H be a subgroup of a group G such that every left coset of H is a right coset of H. Then H is a normal subgroup of G.*
If aH is a left coset of H then $aH = Hb$ for some $b \in G$. Now $a \in aH = Hb \Rightarrow a = hb$ for some $h \in H \Rightarrow ab^{-1} = h \in H \Rightarrow Hb = Ha$. So $aH = Ha$. Thus H is normal in G.

Definition 2.4. *A group G is said to be a simple group if it has no nontrivial normal subgroup.*

Example 2.5. *(i) Every group of prime order is a simple group, since by Lagrange's theorem the only subgroups of a prime order group are $\{e\}$ and the group itself.*
(ii) Any cyclic group of composite order is not simple as in a cyclic group every subgroup is normal and every cyclic group has a nontrivial subgroup, unless it is of prime order.

Theorem 2.6. *Suppose H be a subgroup of a group G of index 2. Then H is a normal subgroup of G.*

Proof. Since $[G : H] = 2$ the only two distinct left cosets of H are H and $G - H$. Similarly the only two distinct right cosets are H and $G - H$. Now for any $a \in G$ $aH = H = Ha$ iff $a \in H$ and $aH = G - H = Ha$ iff $a \notin H$.Hence $aH = Ha$ $\forall a \in G$ showing that H is normal in G. □

Example 2.7. *In the symmetric group S_n the alternating group A_n which the subgroup consisting of all even permutations of S_n contains exactly $n!/2$ elements and so $[S_n : A_n] = 2$. Thus A_n is a normal subgroup of S_n. Thus S_n is a not a simple group for $n > 2$.*

Proposition 2.8. *A subgroup H of a group G is normal in G iff $aHa^{-1} \subseteq H \; \forall a \in G$.*

Proof. If H is normal in G, then for any $a \in G$ and for any $h \in H$ $aha^{-1} = h_1aa^{-1}$ (since $aH = Ha$) $= h_1 \in H$. So $aHa^{-1} \subseteq H$.
Conversely, if $aHa^{-1} \subseteq H \; \forall a \in G$, then for any $a \in G$ $aH = aHa^{-1}a \subseteq Ha$ and $Ha = aa^{-1}Ha \subseteq aH$ implies $aH = Ha \; \forall a \in G$ which implies that H is normal. □

Corollary 2.9. *If H and K are two normal subgroups of a group G then (i) $H \cap K$ is a normal subgroup of G.*
(ii) $HK = KH$ is a normal subgroup of G.
(iii) $< H \cup K >= HK$.

Corollary 2.10. *If $\{H_\alpha : \alpha \in \Gamma\}$ be a family of normal subgroups of a group G then $\bigcap_{\alpha \in \Gamma} H_\alpha$ is a normal subgroup of G.*

Proposition 2.11. *A subgroup H of a group G is normal in G iff $aHa^{-1} = H \; \forall a \in G$.*

Proof. If H is normal in G, then for any $a \in G$ $aHa^{-1} = Haa^{-1} = H$
Conversely, if $aHa^{-1} = H \; \forall a \in G$, then for any $a \in G$ $aH = aHa^{-1}a = Ha$ and which implies that H is normal. □

Corollary 2.12. *Suppose H be a finite subgroup of a group which is the unique subgroup of G of order $|H|$. Then H is a normal subgroup of G.*

Proof. Since H is a subgroup of G, for any $a \in G$ aHa^{-1} is also a subgroup of G with $|aHa^{-1}| = |H|$. So from uniqueness $aHa^{-1} = H$ showing that H is normal in G. □

Example 2.13. *Suppose G be a group of order 51, which has a subgroup of H of order 17. Then H is a normal subgroup of G.*
If K is subgroup of G other than H of order 17 then $|H \cap K| = 1$. Then $|HK| = |H||K|/|H \cap K| > 51$, a contradiction. So H is the unique subgroup of G of order 17. Hence H is a normal subgroup of G.

Theorem 2.14. *Suppose A and B be two normal subgroups of a group G and $A \cap B = \{e\}$. Then $ab = ba \; \forall a \in A, \, b \in B$.*

Proof. From the normality conditions of A and B, $aba^{-1}b^{-1} \in A \cap B = \{e\}$. Hence the result. □

Example 2.15. *Suppose H be a subgroup of G. Then $N(H) = \{x \in G : xHx^{-1} = H\}$ is a subgroup of G and H is a normal subgroup of $N(H)$.*

$N(H)$ is called the normaliser of H in G. It is the maximal subgroup of G containing H in which H is normal.

Example 2.16. *If H is a normal subgroup of G and K is a subgroup of G containing H, then H is a normal subgroup of K.*

Example 2.17. *If H is a normal subgroup of K and K is a normal subgroup of G then H may not be a normal subgroup of G.*
Consider the dihedral group D_4 generated by two elements a, b with $o(a) = 4, o(b) = 2, ba = a^3b$. $D_4 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$. Here $K = \{e, ab, a^2, a^3b\}$ is a normal subgroup (since index of K is 2) of D_4 and $H = \{e, a^3b\}$ is a normal subgroup of K, but H is not a normal subgroup of D_4.

3 Quotient Group

We shall now construct a new group from the original group.

Theorem 3.1. *Suppose H be a normal subgroup of $(G, \cdot)$. Consider the set $\mathscr{L} = \{aH : a \in H\}$ of all distinct left cosets of H in G. Define the binary operation $\cdot$ on $\mathscr{L}$ by $aH \cdot bH = (a \cdot b)H$ $\forall a, b \in G$. Then $\mathscr{L}$ is a group under the operation $\cdot$.*

Proof. We shall first show that the operation is well defined on $\mathscr{L}$. Indeed, if $aH = bH$ and $cH = dH$, then $a^{-1}b \in H$ and $c^{-1}d \in H$. Since H is normal $(ac)^{-1}bd = c^{-1}a^{-1}bd = c^{-1}a^{-1}bcc^{-1}d \in H$ implies $aH \cdot cH = bH \cdot dH$. The defining properties of a group are then obvious. □

Definition 3.2. *The above group $\mathscr{L}$ is called the quotient group of H in G which will be denoted by G/H.*

The following theorem is obvious from the definition.

Theorem 3.3. *If H is a subgroup of an abelian group G then G/H is a quotient group, which is also abelian.*

Theorem 3.4. *If H is a subgroup of a cyclic group G then G/H is a quotient group, which is also cyclic. Further more if a is a generator of G then aH is a generator of G/H.*

Theorem 3.5. *If the quotient group $G/Z(G)$ is cyclic then G is an abelian group.*

Proof. Suppose $G/Z(G)$ is a cyclic group generated by $aZ(G)$. Then any $g \in G$ $gZ(G) = a^mZ(G)$ for some $m \in \mathbb{Z}$. Now for any $g_1, g_2 \in G$ $g_1 = a^mp$ and $g_2 = a^nq$ for some $m, n \in \mathbb{Z}$ and $p, q \in Z(G)$ implies $g_1g_2 = a^mpa^nq = a^ma^npq = a^{m+n}qp = a^{n+m}qp = a^nqa^mp = g_2g_1$. □

4 Group Homomorphisms

We shall now study the category of groups by defining a suitable morphism between group objects, which should be a mapping preserving the binary operation.

Definition 4.1. *Given two groups* $(G, +)$ *and* $(G_1, \cdot)$ *a mapping* $f : G \longrightarrow G_1$ *is said to be a group homomorphism or simply a morphism if* $f(a+b) = f(a) \cdot f(b)$ $\forall a, b \in G$. *If no ambiguity occur we simply write the defining condition as* $f(ab) = f(a)f(b)$ $\forall a, b \in G$.

Example 4.2. *Given any group* G *the identity mapping* $i : G \longrightarrow G$ *is a group homomorphism.*

Example 4.3. *Given any groups* G *and* G_1 *the mapping* $f : G \longrightarrow G_1$ *defined by* $f(g) = e_{G_1}$ $\forall g \in G$ *is a group homomorphism, called the trivial homomorphism.*

Proposition 4.4. *Suppose* $f : G \longrightarrow G_1$ *be a homomorphism. Then*
(i) $f(e_G) = e_{G_1}$
(ii) $f(a^{-1}) = (f(a))^{-1}$ $\forall a \in G$
(iii) $f(a^n) = (f(a))^n$ $\forall n \in \mathbb{Z}$ $\forall a \in G$
(iv) $o(f(a))$ *divides* $o(a)$ $\forall a \in G$ *having finite order.*
(v) the set $Imf = f(G)$=$\{f(a) : \forall a \in G\}$ *is a subgroup of* G_1
(vi) If G *is abelian then* Imf *is also an abelian group.*
(vii) If G *is cyclic then* Imf *is also a cyclic group.*

Proof. (i) $f(e_G) = f(e_G e_G) = f(e_G)f(e_G) \Rightarrow f(e_G) = e_{G_1}$.
(ii) From (i) $e_{G_1} = f(e_G) = f(a)f(a^{-1}) \Rightarrow f(a^{-1}) = (f(a))^{-1}$ $\forall a \in G$
(iii)The result follows immediately from (i) and (ii) and induction on n.
(iv) The result directly follows from (iii).
(v) Since $e_{G_1} = f(e_G)$, Imf is nonempty. Now for any $x = f(a)$, $y = f(b)$ in Imf, $xy^{-1} = f(ab^{-1}) \in Imf$ implies that Imf is a subgroup of G_1.
(vi) $\forall a, b \in G$, $f(a)f(b) = f(ab) = f(ba) = f(b)f(a)$ implies Imf is abelian if G is abelian.
(vii) Follows directly from (iii) □

We often denote the identity elements of two groups G and G_1 by the same symbol e.

Example 4.5. *For any* $n \in \mathbb{N} \cup \{0\}$, *the map* $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ *defined by* $f(a) = na$ $\forall a \in \mathbb{Z}$ *is a group homomorphism. In fact any group homomorphism* $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ *is of the above form.*
Clearly the function $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ *defined by* $f(a) = na$ $\forall a \in \mathbb{Z}$ *is a group homomorphism.*
Now suppose $f : \mathbb{Z} \longrightarrow \mathbb{Z}$ *is a group homomorphism. Let* $f(1) = n$. *Then from the previous proposition(iii), it is clear that* $f(a) = na$ $\forall a \in \mathbb{Z}$.

Definition 4.6. *Given two groups $(G, +)$ and $(G_1, \cdot)$ a homomorphism $f : G \longrightarrow G_1$ is said to be*

(i) a monomorphism if f is injective.

(ii) an epimorphism if f is surjective.

(iii) an isomorphism if f is bijective.

We say that G is isomorphic with G_1 if there is an isomorphism $f : G \longrightarrow G_1$. We denote this by $G \cong G_1$.

Theorem 4.7. *Suppose $f : G \longrightarrow G_1$ is a homomorphism then the set $H = \{a \in G : f(a) = e\}$ is a normal subgroup of G.*

Proof. Since $f(e) = e$, $H \neq \emptyset$. For any $a, b \in H$, $f(ab^{-1}) = f(a)(f(b))^{-1} = e \Rightarrow ab^{-1} \in H$, showing that H is a subgroup of G.
Now for any $a \in G, h \in H$, $f(aha^{-1}) = f(a)e(f(a))^{-1} = e \Rightarrow aha^{-1} \in H$ implies H is a normal subgroup of G. □

The above subgroup is called the kernel of the homomorphism f and it is denoted by $Kerf$.

Theorem 4.8. *Suppose $f : G \longrightarrow G_1$ is a homomorphism then f is a monomorphism iff $Kerf = \{e\}$*

Proof. If f is a monomorphism then, $a \in Kerf \Leftrightarrow f(a) = e \Leftrightarrow a = e$.
Conversely if $Kerf = \{e\}$, then $f(a) = f(b) \Rightarrow f(ab^{-1}) = e \Rightarrow ab^{-1} \in Kerf \Rightarrow a = b \Rightarrow$ f is injective. □

Example 4.9. *Consider the matrix group $GL_n(\mathbb{R})$ of all non singular real matrices of order n with respect to matrix multiplication.*
Here the mapping $f : GL_n(\mathbb{R}) \longrightarrow \mathbb{R}^*$ defined by $f(A) = \det A$ is a group homomorphism with $Kerf = SL_n(\mathbb{R})$, where $SL_n(\mathbb{R})$ is the special linear group consisting of matrices of unit determinant. So from previous theorem $SL_n(\mathbb{R})$ is a normal subgroup of $GL_n(\mathbb{R})$.

Theorem 4.10. *Suppose $f : G \longrightarrow G_1$ is an epimorphism then*

(i) if H is a normal subgroup of G then $f(H)$ is a normal subgroup of G_1.

(ii) if K is a normal subgroup of G_1 then $f^{-1}(K)$ is a normal subgroup of G.

Proof. (i) Since H is a subgroup of G, clearly $f(H)$ is a subgroup of G_1.
Now for any $y \in G_1, x \in f(H)$, $y = f(a)$, $x = f(h)$ for some $a \in G, h \in H$. Then $yxy^{-1} = f(aha^{-1}) \in f(H)$, showing that $f(H)$ is a normal subgroup of G_1.
(ii) Proof is similar to (i). □

Example 4.11. *Suppose* $f : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ *is defined by* $f(a) = [a]$ *for all* $a \in \mathbb{Z}$. *Then* f *is an epimorphism.*

Example 4.12. *Suppose* $\varphi : G \longrightarrow G/H$ *be defined by* $\varphi(a) = aH\ \forall a \in G$. *Then* f *is an epimorphism, called the natural or canonical homomorphism.*

Example 4.13. *Suppose* G *be an abelian group and* $f : G \longrightarrow G$ *is defined by* $f(a) = a^2$ *for all* $a \in G$. *Then* f *is a homomorphism. Also if order of* G *is odd then* f *is an isomorphism.*

Example 4.14. *Suppose* G *be an abelian group of oredr* n *and* $f : G \longrightarrow G$ *is defined by* $f(a) = a^m$ *for all* $a \in G$. *Then* f *is is an isomorphism iff* $\gcd(m, n) = 1$.

Example 4.15. *Suppose* G *be a group and* $f : G \longrightarrow G$ *is defined by* $f(a) = a^{-1}$ *for all* $a \in G$. *Then* f *is an isomorphism iff* G *is abelian.*

Theorem 4.16. *Suppose* $f : G \longrightarrow G_1$ *is an isomorphism then*
(i) G *is abelian iff* G_1 *is abelian.*
(ii) G *is cyclic iff* G_1 *is cyclic.*
(iii) $o(a) = o(f(a))$ *for all* $a \in G$.
(iv) if H *is a normal subgroup of* G *iff* $f(H)$ *is a normal subgroup of* G_1.
(v) $f^{-1} : G_1 \longrightarrow G$ *is an isomorphism.*

Proof. The proof is straight forward □

Theorem 4.17. *Suppose* $f : G \longrightarrow G_1$ *and* $g : G_1 \longrightarrow G_2$ *are homomorphism then* gof *is a homomorphism.*

Proof. The proof is straight forward. □

5 Automorphism group

Definition 5.1. *An isomorphism* $f : G \longrightarrow G$ *is called an automorphism.*

Example 5.2. *If* G *is a gorup of order* 7, *then* $f : G \longrightarrow G$ *defined by* $f(x) = x^2\ \forall x \in G$ *is an automorphism.*

Theorem 5.3. *Suppose* G *be a group. Then the set* S *of all isomorphism on* G *forms a group under mapping composition.*

Proof. Since the identity mapping is an isomorphism, $S \neq \emptyset$. Suppose $f, g \in S$. Then f and g being isomorphisms, fog is also an isomorphism. Also f^{-1} is an isomorphism. So S is a group. □

This group is called the automorphism group of G, denoted by $Aut(G)$.

Theorem 5.4. *The automorphism group of $\mathbb{Z}_n$ is isomrphic with U_n.*

Proof. Suppose $f : \mathbb{Z}_n \longrightarrow \mathbb{Z}_n$ be an automorphism. Then $f([1]) = [n_f]$ is a generator of $\mathbb{Z}_n$. Then $\gcd(n_f, n) = 1$. So $[n_f] \in U_n$. Define $\varphi : Aut(\mathbb{Z}_n) \longrightarrow U_n$ by $\varphi(f) = [n_f]\ \forall f \in Aut(\mathbb{Z}_n)$. Then it is a routine matter to check that $\varphi : Aut(\mathbb{Z}_n) \longrightarrow U_n$ is an isomorphism. □

Theorem 5.5. *Suppose $x \in G$. Define $f_x : G \longrightarrow G$ by $f_x(a) = xax^{-1}\ \forall a \in G$. Then*
(i) f_x is an automorphism.
(ii) $f_x o f_y = f_{xy}\ \forall x, y \in G$.
(iii) ${f_x}^{-1} = f_{x^{-1}}$
(iv) for any $\varphi \in Aut(G)$ $\varphi o f_x o \varphi^{-1} = f_{\varphi(x)}$
(v) The set $\{f_x : G \longrightarrow G / f_x(a) = xax^{-1} \forall a \in G\}$ is a normal subgroup of $Aut(G)$

Proof. (i) For $a, b \in G$, $f_x(ab) = xabx^{-1} = xax^{-1}xbx^{-1} = f_x(a)f_x(b)$ implies f_x is a homomorphism. Verification of isomorphism is trivial.
(ii) $(f_x o f_y)(a) = f_x(yay^{-1}) = x(yay^{-1})x^{-1} = xy(a)(xy)^{-1} = f_{xy}(a)\ \forall a \in G$.
(iii) $f_x o f_{x^{-1}} = f_{xx^{-1}} = f_e$= identity mapping.
(iv) for any $\varphi \in Aut(G)$ $(\varphi o f_x o \varphi^{-1})(a) = (\varphi o f_x)(\varphi^{-1})(a) = (\varphi o f_x)(b)$ [where $(\varphi)(b) = a$] $= \varphi(xbx^{-1}) = \varphi(x)a\varphi(x)^{-1} = f_{\varphi(x)}(a)$
(v) is straightforward from (i) to (iv). □

The set $\{f_x : G \longrightarrow G / f_x(a) = xax^{-1} \forall a \in G\}$ is called the group of inner automorphisms, denoted by $InnG$.

6 Isomorpism Theorems and Their Applications

In this section we establish the relation between homomorphic image of a group and a quotient group.

Theorem 6.1. *Suppose $f : G \longrightarrow G_1$ be an epimorphism and H be a normal subgroup of G contained in $Kerf$. Suppose $g : G \longrightarrow G/H$ be the natural homomorphism. Then there exists a unique epimorphism $\varphi : G/H \longrightarrow G_1$ such that $f = \varphi o g$. Furthermore, φ is an isomorphism iff $H = Kerf$*

Proof. If $b \in aH$, then $b = ah$, for some $h \in H$ and $f(b) = f(ah) = f(a)$. Therefore, f has the same effect on every element of the coset aH. Thus the mapping $\varphi : G/H \longrightarrow G_1$ defined by

$\varphi(aH) = f(a)$ is well defined. Since $\varphi(aHbH) = \varphi(abH) = f(ab) = f(a)f(b) = \varphi(aH)\varphi(bH)$, φ is a homomorphism.Also $Im\varphi = Imf$.From the definition $f = \varphi og$.

Now $aH \in Ker\varphi \Longleftrightarrow f(a) = e \Longleftrightarrow a \in Kerf$. So $Ker\varphi = (Kerf)/H$. Therfore, φ is an isomorphism iff $Kerf = H$.

Finally φ is unique since it is completely determined by f. □

From the above theorem we see that every homomorphism of a group G onto G_1 induces an isomorphism of $G/Kerf$ onto G_1. This result plays a fundamental role in group theory to classify the groups upto isomorphism.It is known as Fundamental theorem or first isomorphism theorem for groups.

Theorem 6.2. *Suppose $f : G \longrightarrow G_1$ be a homomorphism then $G/Kerf$ is isomorphic to $f(G)$.*

Proof. Here $f : G \longrightarrow f(G)$ is an epimorphism. So considering $H = Kerf$ the result follows from the previous theorem. □

Example 6.3. *For any $n \in \mathbb{N}$, $S_n/A_n \cong \mathbb{Z}_2$ as the mapping $f : S_n \longrightarrow Z_2$ defined by $f(x) = 0$ if x is an even permutation, $f(x) = 1$ if x is an odd permutation is an onto homomorphism with $Kerf = A_n$.*

Example 6.4. *The group $\mathbb{R}/\mathbb{Z}$ is isomorphic with the circle group $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Here the mapping $f : \mathbb{R} \longrightarrow S^1$ defined by $f(x) = exp(2\pi ix)\forall x \in R$ is an epimorphism with $Kerf = \mathbb{Z}$.*

Example 6.5. *For any group G the quotient group $G/Z(G)$ is isomorphic to $InnG$. Here the mapping $f : G \longrightarrow InnG$ defined by $f(x) = f_x$ [as defined in theorem 5.5] is an epimorphism with $Kerf = Z(G)$.*

Example 6.6. *The group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$ follows form the first isomorphism theorem applied on the epimorphism $f : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ defined by $f(a) = [a]$.*

Example 6.7. *Consider the mapping $f : GL_2(\mathbb{Z}_p) \longrightarrow (\mathbb{Z}_p)^*$ defind by $f(A) = \det A$. Then f is an epimorphism with $Kerf = SL_2(\mathbb{Z}_p)$. By First isomorphism Theorem $GL_2(\mathbb{Z}_p)/SL_2(\mathbb{Z}_p) \cong (\mathbb{Z}_p)^*$.*

Theorem 6.8. *Suppose K and N are subgroups of a group G and N is normal in G. Then $K/N \cap K$ is isomorphic to NK/N.*

Proof. Let us define a mapping f from K to NK/N by $f(x) = xN$. Then f is an epimorphism. By First isomorphism theorem the result follows. □

Corollary 6.9. *Suppose K and H are normal subgroups of a group G and $K \subseteq H$. Then H/K is a normal subgroup of G/K and $(G/K)/(H/K)$isomorphic to G/H.*

Example 6.10. *From the above corollary $\mathbb{Z}/ < 3 >$ is isomorphic to $(\mathbb{Z}/ < 6 >)/(< 3 > / < 6 >)$.*

Theorem 6.11. *Suppose $f : G \longrightarrow G_1$ be an epimorphism then the assignment $K \longmapsto f(K)$ defines a one-one correspondence between the set $\mathscr{H}$ of all subgroups of G containing $Kerf$ and the set $f(\mathscr{H})$ of all subgroups of G_1. Also under this map normal subgroups correspond to normal subgroup.*

Example 6.12. *(i)The groups $(\mathbb{R}, +)$ and $(\mathbb{Z}+)$ are not isomorphic as $(\mathbb{Z}+)$ is cyclic and $(\mathbb{R}, +)$ is non cyclic.*
(iii) The groups $(\mathbb{R}, +)$ and $(\mathbb{Q}, +)$ are not isomorphic although both of them are non cyclic.

Example 6.13. *The groups $S_3, \cdot$ and $\mathbb{Z}_6, +)$ are not isomorphic as $S_3, \cdot$ is non abelian but $\mathbb{Z}_6, +)$ is abelian.*

Example 6.14. *The groups D_4 and Q_8 are non isomorphic as D_4 contains only two elements of order 4, while Q_8 contains exactly six elements of order 4.*

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