E LEARNING MODULE

GENERAL TOPOLOGY II

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1 Compactness

Recall that a topological space (X, τ) is a nonempty set X together with a subset $\tau \subset P(X)$ which satisfy (a) ϕ and X are in τ, and (b) is closed under countable union and finite intersection. The members of τ are called open sets and complement of each open set is called a closed set in the toplogical space.

In a topological space (X, τ) a subset $N$is called a neighbourhood of a point $a \in X$ if there is an open set $U \in \tau$ such that $a \in U \subseteq N$ and then a is called an interior point of N. The set of all interior points of A in a topological space X is called interior of A and it is denoted by $int_X A$. In a topological space (X, τ) a point $a \in X$ is called a limit point or accumulation point or cluster point of a subset N if for every open set $U \in \tau$ such that $a \in U$ we have $U \cap N$ contains atleast one point other than a. The set of all cluster points of A in a topological space X is called derived set of A and it is denoted by A^d. The set $A \cup A^d$ is called the closure of A, denoted by $cl_X A$. In a topological space (X, τ) $int_X A$ is the largest open set contained in A and it is the union of open sets which are contained in A and $cl_X A$ is the smallest closed set containing A and it is the intersection of all closed sets containg A. Given a topological space (X, τ) a family $\mathscr{B}$ of open sets is a base for the topology if any open set can be expressed as the union of some members of the family $\mathscr{B}$. If there is a countable base then the space is called second countable. A space is called first countable if every point $x \in X$ has a countable neighbourhood base. The space (X, τ) is called separable if it has a countable dense subset.The readers are familiar with the basic properties of first countable, separable and second countable properties.

A function between two topological spaces is called continuous if the inverse image of any open set in the co domain space is an open set in the domain space. A bijective continuous map between topological spaces is called a homeomorphism if the inverse function is also continuous and then two spaces are said to be homeomorphic. Any property which is preserved by the homeomorphism is called a topological properties.In first semester course we have studied some topological properties like T_0, T_1, Hausdorff, regularity, normal, completely regular, completely normal properties. In this section we shall discuss about the compactness property of a space.

Definition 1.1. *A family of subsets $\{G_\alpha : \alpha \in \Gamma\}$ of $X$is said to be a cover of X if $X = \bigcup_{\alpha \in \Gamma} G_\alpha$. If all G_α are open sets then the family is called an open cover of X. A sub family $\{G_\alpha : \alpha \in \Gamma_1\}$of an open cover $\{G_\alpha : \alpha \in \Gamma\}$ is said to be a sub cover if the sub family $\{G_\alpha : \alpha \in \Gamma_1\}$ is itself an open cover of X. A topological space X is said to be compact if every open cover of X has a finite sub cover. A subset Y of a topological space X is said to be compact if the topological subspace Y is compact.*

Example 1.2. *Finite subset of any topological space is compact.*

Example 1.3. *A discrete space X is compact if and only if X is a finite set.*

Example 1.4. *Any infinite set X with cofinite topology τ is a compact space. Indeed if $\{G_\alpha : \alpha \in \Gamma\}$ is an open cover of X then for any $\alpha_0 \in \Gamma$ with $G_{\alpha_0} \neq \phi$, from the definition of cofinite topology, $(G_{\alpha_0})^c$ is a finite set F. For each $i \in F$ there is an $\alpha_i \in \Gamma$ such that $i \in G_{\alpha_i}$. Clearly then $\{G_{\alpha_i} : i \in F\} \bigcup G_{\alpha_0}$ is a finite subcover of $\{G_\alpha : \alpha \in \Gamma\}$.*

Example 1.5. *The space $\mathbb{R}$ with the usual topology is not compact as the family $\{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ having no finite subcover.*

Theorem 1.6. *Any closed subspace of a compact space is compact.*

Proof. Suppose X be a compact space and Y be a closed subset of X. Let $\{G_\alpha : \alpha \in \Gamma\}$ be an open cover of Y. Then $\{G_\alpha : \alpha \in \Gamma\} \cup \{Y^{\complement}\}$ is an open cover of X. X being compact, there is a finite subcollection $\{G_\alpha : \alpha \in \Gamma_1\}$ of $\{G_\alpha : \alpha \in \Gamma\}$ for some finite subset $\Gamma_1 \subseteq \Gamma$ such that $\{G_\alpha : \alpha \in \Gamma_1\} \cup \{Y^{\complement}\}$ covers X. Then $\{G_\alpha : \alpha \in \Gamma_1\}$ covers Y shows that Y is compact. $\square$

Theorem 1.7. *Any compact subset of a Hausdorff space is closed.*

Proof. Suppose Y be a compact subset of a Hausdorff space X.
To prove that Y is closed it is enough to show that outside Y there is no limit point of Y.
For any $x \in Y^{\complement}$ and for any $y \in Y$ using Hausdorff property there are disjoint open sets $U(y)$ and $V(y)$ in X such that $x \in U(y)$ and $y \in V(y)$.
Since Y is compact, the open cover $\{V(y) : y \in Y\}$ has a finite subcover, say $\{V(y) : y \in F\}$ for some finite subset $F \subseteq Y$.
Suppose $U = \bigcap_{y \in F} U(y)$. Then U is an open neighbourhood of x which is disjoint with $\bigcup_{y \in F} V(y)$ and so $U \cap Y = \emptyset$.
So x is not a limit point of Y. Hence Y is closed in X. $\square$

Theorem 1.8. *Continuous image of a compact space is compact.*

Proof. Let $f : X \longrightarrow Y$ be a continuous mapping between two topological spaces X and Y and let X be compact.
We shall show that $f(X)$ is a compact subset of Y.
Suppose $\{U_\alpha : \alpha \in \Gamma\}$ be an open covering of $f(X)$. f being continuous, $\{f^{-1}(U_\alpha) : \alpha \in \Gamma\}$ is an open cover of X. Since X is compact, there is a finite subset $\Gamma_1 \subseteq \Gamma$ such that $\{f^{-1}(U_\alpha) : \alpha \in \Gamma_1\}$ is an open cover of X. Then $\{U_\alpha : \alpha \in \Gamma_1\}$ covers $f(X)$, showing that $f(X)$ is compact. $\square$

Theorem 1.9. *If $f : X \longrightarrow Y$ is a continuous bijection from a compact topological spaces X onto a Hausdorff space Y then f is a homeomorphism.*

Proof. To prove this we only need to show that $f^{-1} : Y \longrightarrow X$ is continuous. Suppose K be a closed subset of X. Then K is compact. f being continuous $f(K)$ is compact in Y. Since Y is Hausdorff $f(K)$ is closed in Y, showing that $f^{-1} : Y \longrightarrow X$ is continuous. □

The following theorem known as Tychonoff theorem on product space can be proved using properties of ultrafilters.

Theorem 1.10. *Product of arbitrary family of topological space is compact iff each factor space is compact.*

Example 1.11. *In $\mathbb{R}$ with usual topology, a subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*
This is the Heine -Borel Property, we have studied in undergraduate Real Analysis.
Here we can prove this result in more general form: a subset F of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.
We first prove that $I = [0, 1]$ is compact.
Suppose $\{G_\gamma : \gamma \in \Gamma\}$ be an open cover of I and let K be the set of all points $c \in I$ such that some finite subcollection from $\{G_\gamma : \gamma \in \Gamma\}$ covers $[0, c]$. Clearly $0 \in K$. Also, if $c \in K$ and $b \leq c$ then $b \in K$. Thus K is a sub interval of I containing 0. Moreover, if $c \in K$, then any finite subcollection from $\{G_\gamma : \gamma \in \Gamma\}$ which covers $[0, c]$ also covers $[0, c + \epsilon$ for some $\epsilon > 0$ (unless $c = 1$, in which case we have finished). Thus K is an open set in I. Finally if k is the right end point of K, then $k \in K$, for pick $G_i \in \{G_\gamma : \gamma \in \Gamma\}$ such that kG_i. Then $(k - \epsilon, k] \subseteq G_i$ for some $\epsilon > 0$ so that by adding G_i to a finite subcollection from $\{G_\gamma : \gamma \in \Gamma\}$ which covers $[0, k - \epsilon]$ we obtain a finite subcollection from $\{G_\gamma : \gamma \in \Gamma\}$ which covers $[0, k]$. Now K is a closed subinterval of I which contains 0 and an open set in I. Thus $K = I$. This proves that I is compact.
Suppose F be a closed and bounded subset of $\mathbb{R}^n$. Then K is a closed subset of an $n - fold$ product $[-c, c] \times [-c, c] \times \ldots \times [-c, c]$ of intervals for some $c \in \mathbb{R}$. Now each interval $[-c, c]$ is homeomorphic with $[0, 1]$ which is compact. So each such $n - fold$ is compact, by Tychonoff theorem. K being a closed subset of this $n - fold$ it is compact.

Definition 1.12. *A family of subsets $\{F_\alpha : \alpha \in \Lambda\}$ is said to have finite intersection property if intersection of any finite subfamily is nonvoid.*

Example 1.13. *The family $\{[n, \infty] : n \in \mathbb{N}\}$ has finite intersection property, but $\bigcap_{n \in \mathbb{N}} [n, \infty] = \emptyset$*

Theorem 1.14. *A topological sapce X is compact iff any family of closed sets in X having finite intersection property(FIP) has nonemty intersection.*

Proof. First we note that a family $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of closed sets in X having void intersection iff (using De Morgan's law) $\bigcup_{\alpha\in\Lambda} F_\alpha = X$ iff the family $\{\complement F_\alpha : \alpha \in \Lambda\}$ is an open cover of X.
Thus if the family $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ of closed sets in X has empty intersection then by compactness of X $\mathscr{F}$ has a finite subfamily which has also empty intersection proves necessary part of the theorem.
For sufficiency suppose, $\mathscr{G} = \{G_\alpha : \alpha \in \Lambda\}$ be an open cover of X. Then $\mathscr{F} = \{F_\alpha : \alpha \in \Lambda\}$ (where $F_\alpha = \complement G_\alpha$) is a family of closed sets having empty intersection. So $\mathscr{F}$ has a finite subfamily which has also empty intersection. Which again by De Morgan's law implies $\mathscr{G}$ has a finite subcover, showing that X is compact. □

In discussing continuity of a function we saw that although it requires that the inverse image of every open set be open, it actually suffices if the inverse image of every member of some sub base for the codamain is open.
It is natural to research something similar to happen for compactness. The outcome is affirmative due to the Alexander Sub- base Theorem:

Theorem 1.15. *A topological space X is compact iff it has a base $\mathscr{B}$ with the property that every cover of X by members of $\mathscr{B}$ has a finite subcover.*

Proof. The necessity of this theorem is obvious. For the sufficiency, assume that $\mathscr{B}$ is a base for X with the property that every cover of X by members of $\mathscr{B}$ has a finite subcover.
Suppose $\mathscr{U}$ be any open cover of X. For each $U \in \mathscr{U}$ there is a subfamily $\mathscr{B}_U$ of $\mathscr{B}$ such that $U = \bigcup_{B\in\mathscr{B}_U} B$. Then $\mathscr{V} = \bigcup_{U\in\mathscr{U}} \mathscr{B}_U$ is an open cover of X. Moreover $\mathscr{V} \subseteq \mathscr{B}$. So $\mathscr{V}$ has a finite subcover say, $\{V_1, V_2, \ldots, V_n\}$. For each $i \in \{1, 2, \ldots, n\}$ there exists $U_i \in \mathscr{U}$ such that $V_i \in \mathscr{B}_{U_i}$. But then clearly $V_i \subseteq U_i$ and so $\{U_1, U_2, \ldots, U_n\}$ is a finite subcover of $\mathscr{U}$ showing that X is compact. □

Definition 1.16. *Let X be a topological space and $\mathscr{C}$ a family of closed subsets of X. Then $\mathscr{C}$ is said to be a closed base (closed sub-base) for X if the family of complements of members of $\mathscr{C}$ is a base (sub-base) for X.*

Theorem 1.17. *A family $\mathscr{C}$ of closed subsets of X is a closed sub-base for X iff the family consisting of all finite unions of members of $\mathscr{C}$ is a closed base for X.*

Proof. This result follows trivially from De Morgan's law and the previous definition. □

Theorem 1.18. *A topological space X is compact iff it has a closed base $\mathscr{C}$ with the property that every subfamily of $\mathscr{C}$ having the finite intersection property has a nonempty intersection.*

Proof. The proof is immediate from the previous two theorems and De Morgan's law. □

Theorem 1.19. *A topological space X is compact iff there exists a closed sub-base $\mathscr{C}$ for X such that every subfamily of $\mathscr{C}$ having the finite intersection property has a nonempty intersection.*

Proof. The necessity of the condition is trivial.
For sufficiency, suppose $\mathscr{C}$ is a closed sub-base for the space X which has the given property. Let $\mathscr{D}$ be the family of all finite unions of members of $\mathscr{C}$. Then by theorem 1.16 $\mathscr{D}$ is a closed base for X. In view of theorem 1.17 it is enough to show that every subfamily of $\mathscr{D}$ having finite intersection property has a nonempty intersection.
Suppose $\mathscr{F}$ be a subfamily of $\mathscr{D}$ having the $f.i.p.$ We have to show that $\bigcap_{F\in\mathscr{F}} F \neq \emptyset$.
Using Zorn's lemma, we can find a subfamily $\mathscr{E}$ of $\mathscr{D}$ such that $\mathscr{F} \subseteq \mathscr{E}$ and $\mathscr{E}$ is maximal with respect to $f.i.p.$

□

Corollary 1.20. *A topological space X is compact iff it has a sub-base $\mathscr{S}$ with the property that every cover of X by members of $\mathscr{S}$ has a finite subcover.*

Proof. This follows from theorem 1.16 directly by using De Morgan's law of complementation.

□

Theorem 1.21. *Let X be a Hausdorff space K be a compact subset of X not containing the point $x \in X$. Then x and K are separated by open sets in X*

Proof. For any $y \in K$ using Hausdorff property there are disjoint open sets $U(y)$ and $V(y)$ in X such that $x \in U(y)$ and $y \in V(y)$. Since K is compact, the open cover $\{V(y) : y \in K\}$ has a finite subcover, say $\{V(y) : y \in F\}$ for some finite subset $F \subseteq K$.
Suppose $U = \bigcap_{y\in F} U(y)$. Then U is an open set containing x which is disjoint with $V = \bigcup_{y\in F} V(y)$, an open set containing K. □

Theorem 1.22. *Any compact Hausdorff space is normal and hence a T_4 space.*

Proof. Suppose A and B are two disjoint closed subsets of a compact Hausdorff space X. Then B is compact. Now for any $a \in A$ using theorem 1.21 there are disjoint open sets $U(a)$ and $V(a)$ in X such that $a \in U(a)$ and $B \subseteq V(a)$. Now the family $\{U(a) : a \in A\}$ is an open cover of the compact set A (since A is closed in the compact space X). So there is a finite subset $A_1 \subseteq A$ such that $\{U(a) : a \in A_1\}$ is a finite subcover. Let $U = \bigcup_{a \in A_1} U(a)$ and $V = \bigcap_{a \in A_1} V(a)$. Then both U and V are open sets containing A and B respectively, showing that X is normal. $\square$

Definition 1.23. *A topological space X is said to be $\sigma-compact$ if every countable open cover of X has a finite sub cover. A topological space X is said to be Lindelof if every open cover of X has a countable sub cover.*

From the definitions it is clear that every compact space is both $\sigma-$ *compact* and Lindelof. On the other hand any Lindelof and $\sigma-$ *compact* space is compact.

Definition 1.24. *A topological space X is said to be limit point compact or Frechet compact if every infinite subset of X has a limit point. A topological space X is said to be sequentially compact if every sequence of X has a convergent subsequence.*
From the definition, it is clear that every compact space is Frechet compact.
Indeed if a subset $A \subseteq X$ has no limit point then all the points of A are isolated points and A is closed in the compact space X. So the open covering of the compact subset A by each singleton set consisting of points of A can not have a finite subcover unless the set A is finite.
However a Frechet space may not be a compact space.

Example 1.25. *Suppose Y be a two points set with indiscrete topology. Now the space $X = \mathbb{N} \times Y$ is a Frechet compact space as every nonempty subset of X has a limit point. But X is not compact as the open covering $\{\{n\} \times Y : n \in \mathbb{N}\}$ has no finite subcovering.*

We shall state the equivalence of these three compactness for a particular type of topological spaces

Theorem 1.26. *For a metrizable topological space the following are equivalent:*
(1) *X is a compact space.*
(2) *X is Frechet compact.*
(3) *X is sequentially compact.*

2 Connectedess

The topological study of connectedness is heavily geometric. Connectedness-like properties play an important role in most topological characterization theorems, as well as the study of obstructions to the extension of functions. The use of path connectedness to associate a topological sapce with a group will be studied in Algebraic Topology. The famous Intermediate

Value Property of a continuous real valued function on an interval can be proved in a simplest way by considering the continuous image of a connected space is connected.

Definition 2.1. *Given a topological space (X, τ) a pair (H, K) of subsets of X is called a separation of X if both H and K are nonempty open subsets of X, $X = H \cup K$ and $H \cap K = \emptyset$.*
A topological space X is said to be disconnected if X has a separation by nonempty disjoint open sets. X is said to be connected if it is not disconnected.
A subset Y of the space X is called connected(disconnected) if it is connected(disconnected) as a subspace of of X.

Theorem 2.2. *In a topological space X a pair (U, V) of open sets is a separation of $X iff X = U \cup V$, $U \neq \emptyset, V \neq \emptyset, U \cap cl_X(V) = cl_X(U) \cap V = \emptyset$.*

Proof. Suppose (U, V) is a separation of X. Then $X = U \cup V, U \neq \emptyset, V \neq \emptyset$. Also $U \cap V = \emptyset$ $\Rightarrow U$ is a subset of the closed set $X - V$. $\Rightarrow cl_X(U) \subseteq X - V$. So $cl_X(U) \cap V = \emptyset$ .Similarly $cl_X(V) \cap U = \emptyset$. The converse is trivial. □

Theorem 2.3. *A topological space X is connected if and only if it has no nontrivial subset which is both open and closed.*

Proof. Suppose X is disconnected. Then $X = U \cup V, U \cap V = \emptyset$ for some nonempty open sets U and V. Evidently $U = X - V$ is a closed set implies that X has a nontrivial clopen set.
Conversely if X has a nontrivial clopen subset say, U then U and $X - U$ both are nontrivial open sets which is a separation of X, showing that X is a disconnected space. □

Example 2.4. *Any set X with more than one element is a disconnected space with respect to discrete topology. Hence the set $\mathbb{N}$ of natural numbers with the subspace topology of $\mathbb{R}$ is disconnected.*

Theorem 2.5. *Suppose Xbe a topological space and $A \subseteq B \subseteq cl_X(A)$. If A is a connected subset of X then B is also connected .*
Hence closure of any connected subset is also a connected subset.

Proof. It is enough to show that $cl_X(A)$ is connected(since if $A \subseteq B \subseteq cl_X(A)$ then $B = cl_B(A)$ and we can replace $XbyB$). Suppose $cl_X(A) = U \cup V$ where $UandV$ are disjoint nonempty open sets in $cl_X(A)$. Then $A = (U \cap A) \cup (V \cap A)$ and the latter are disjoint nonempty open sets in A . Thus if $cl_X(A)$ is disconnected, so is A. □

Theorem 2.6. *The only connected subsets of $\mathbb{R}$(with respect to usual topology) are intervals(both proper and improper).*

Proof. Suppose K be a connected subset of $\mathbb{R}$ containing more than one point. If $x, y \in K$ and $x < y$,and if $z \in \mathbb{R}$ such that $x < z < y$ we must show that $z \in K$. For if $z \notin K$ then the pair (U, V) where $U = (-\infty, z) \cap K$ and $V = (z, \infty) \cap K$ is a separation of K showing that K is disconnected. Thus K is an interval.

Conversely if K is an interval in view of theorem 2.5 it is enough to show that K is connected if K is a closed bounded interval. Suppose $K = [0, 1]$, with $K = U \cup V$ where $U and V$ are nonempty disjoint open sets in K and $0 \in U$.

U being an open set some open neighbourhood of 0 is contained in U. So $c = inf(V)$ can not be 0. Now either $c \in U$ or $c \notin V$, and so there is a neighbourhood of c which is contained either in U or in V. But any neighbourhood of c contains a point of U (to the left of c) and a point of V (to the right of c), a contradiction. So K is connected. □

Theorem 2.7. *Continuous image of any connected space is connected.*

Proof. Suppose X be a connected space and $f : X \longrightarrow Y$ is a continuous surjective map. To show Y is connected we see that if U and V are nonempty disjoint open subsets of Y with $Y = U \cup V$ then from the continuity of f $f^{-1}(U)$ and $f^{-1}(V)$ is a separation of X by nonempty disjoint open sets, showing that X is disconnected. Thus Y is a connected topological space. □

Corollary 2.8. *If $f : [a, b] \longrightarrow \mathbb{R}$ is a continuous function and $f(a) \neq f(b)$ then f assumes every values between $f(a)$ and $f(b)$.*

Proof. Since f is continuous and $[a, b]$ is a connected subset of $\mathbb{R}$, from theorem 2.7 $f([a, b])$ is connected subset of $\mathbb{R}$ and so it must be an interval. Hence the result. □

Theorem 2.9. *Any nonempty product of topological spaces is connected iff each factor of the product is connected.*

Remark 2.10. *One of the central problems of topology is to discover necessary and sufficient conditions for homeomorphism between to spaces. There are two sides to this problem - the condition that the spaces should be homeomorphic and the condition that they should not. Little progress has been made on the positive side beyond the classification of surface ($2 - dimensionalmanifolds$), but any topologically invariant property is a means of shewing that certain pair of spaces are not homeomorphic, for if X has the property and Y has not then X and Y are not homeomorphic.From the previous theorem it is clear that connectedness of a space is a topological property. Thus to check whether two topological spaces are homeomorphic or not connectedness property will be a necessary tool in the sense that if one space is* *connected and the other is disconnected then they are not topologically same i.e. they are not homeomorphic.*

Theorem 2.11. *If $n > 1$ then the connectedness of $\mathbb{R}^n$ is not destroyed by removing an enumerable set of points E.*

Example 2.12. *The spaces $\mathbb{R}$ and $\mathbb{R}^n$ are not homeomorphic for $n > 1$.*
Infact the subspace by removing one point from the space $\mathbb{R}$ will be a disconnected space while the subspace by removing one point from $\mathbb{R}^n$ is still connected.

Example 2.13. *The space $[0, \infty]$ is not homeomorphic with $\mathbb{R}$.*

Theorem 2.14. *If A and B are connected subsets of a space X and if $A \cap B \neq \emptyset$ then $A \cup B$ is also connected.*

Proof. Suppose U and V are disjoint open subsets of $A \cup B$ such that $A \cup B = U \cup V$. Let $U = P \cap (A \cup B)$ and $V = Q \cap (A \cup B)$ for some open sets P and Q of X. Then $P \cup Q \supseteq A \cup B$. Suppose $c \in A \cap B$ and let $c \in P \cap A$. Then A being connected and $A = (P \cap A) \cup (Q \cap A)$ we must have $Q \cap A = \emptyset$. Similarly using connectedness of B $Q \cap B = \emptyset$. Then $Q \cap (A \cup B) = \emptyset$. So $A \cup B$ is connected. □

The above theorem can be proved for arbitrary union also:

Theorem 2.15. *If $\{A_\alpha : \alpha \in \Lambda\}$ be a family of connected subsets of X such that $\bigcap_{\alpha\in\Lambda} A_\alpha \neq \emptyset$.Then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is also connected.*

Theorem 2.16. *If Y is a connected subset of X and $\{A_\alpha : \alpha \in \Lambda\}$ be a family of connected subsets all meeting Y, then $\bigcap_{\alpha\in\Lambda} A_\alpha$ is also connected.*

Definition 2.17. *Suppose X be a topological space. A subset $A \subseteq X$ is said to be path connected if for any two point $a, b \in X$, there is a continuous function $f : [0, 1] \longrightarrow X$ such that $f([0, 1]) \subseteq A$ and $f(0) = a, f(1) = b$.*

Theorem 2.18. *Every path connected space is connected.*

Proof. Suppose X be path connected. If possible let X be disconnected. Then there is a nontrivial clopen subset U of X. Suppose $p \in U$, $q \notin U$. Then there is a continuous function $f : [0, 1] \longrightarrow X$ such that $f(0) = p$, $f(1) = q$. Since f is continuous, $f^{-1}(U)$ and $f^{-1}(U^)$ are disjoint nonempty open sets forms a separation of $[0, 1]$ which contradicts the fact that $[0, 1]$ is connected. So X is connected. □

Example 2.19. *The topologists sine curve Γ is connected but not path connected.*

Here $\Gamma = \{(x, 0) : x \leq 0\} \cup \{(x, \sin(\frac{1}{x})) : x > 0)$ is the graph of the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ \sin(\frac{1}{x}) & \text{if } x > 0 \end{cases}$.

The set Γ is not path connected as the points $(0, 0)$ and $(\frac{1}{\pi}, 0)$ are in Γ but there is no continuous function $f : [0, 1] \longrightarrow \Gamma$ such that $f(0) = (0, 0)$, and $f(1) = (\frac{1}{\pi}, 0)$.

Suppose $\Gamma_1 = \{(x, \sin(\frac{1}{x})) : x > 0\}$ and $A = \{(x, \sin(\frac{1}{x})) : x > 0\} \cup \{(0, 0)\}$. Here Γ_1 is path connected and hence connected and $\Gamma_1 A \subseteq cl_{\mathbb{R}^2}(\Gamma_1)$. Hence A is connected. Also A and $(x, 0) : x \leq 0\}$ has nonempty intersection implies $\Gamma = A \cup (x, 0) : x \leq 0\}$ is connected.

Definition 2.20. *Given a topological space X and a point xX, the largest connected subset C_x of X containing x is called the component of x.*

Remark 2.21. *In view of theorem 2.14 C_x being just the union of all connected subset of X containing x*

Also the components make a partition of X. Indeed if for $x \neq y \in X$, then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$; otherwise $C_x \cup C_y$ would be a connected set containing x and y and larger than C_x or C_y, which is impossible.

Theorem 2.22. *The components of X are closed sets.*

Proof. If C is the component of x in X, then $cl_X C$ is a connected set containing x and thus $cl_X C \subseteq C$, showing that C is a closed set. □

Example 2.23. *(a) The Sorgenfrey line ($\mathbb{R}$ with lower limit topology) is not connected.*

(b) Any infinite set with the cofinite topology is connected.

(c) A countable subset C of $\mathbb{R}$ is connected iff C is a singleton set.

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