Lecture Notes on Metric Spaces

M.Sc Sem 1 (August,2019- January,2020)

(Under the Curriculum of M.Sc in Pure Mathemtics, Department of Pure Mathematics, University of Calcutta)

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CONTENTS

1. Metric Spaces: Introductory Concepts

1.1 Definition: A metric space is an ordered pair (X, d) where X is any nonempty set and $d: X x X \to \mathbb{R}_{\geq 0}$ is a mapping satisfying

a) $d(x, y) = 0$ if and only if $x = y$.

b) $d(x, y) = d(y, x)$ for all x, y in X.

c) $d(x, y) + d(y, z) \geq d(x, z)$ for all x, y, z in X.

The mapping $d: X x X \to \mathbb{R}_{\geq 0}$ satisfying a), b), c) is called a metric on X.

If there is no ambiguity over the metric d then simply we call X is a metric space.

A mapping $d: X x X \to \mathbb{R}_{\geq 0}$ satisfying b), c), and a'): d(x,x) = 0 for all x is called a pseudometric on X and then (X,d) is called a pseudometric space.

1.2 Examples: Some familiar examples of metric spaces are

1. $\mathbb{R}^n$ with Euclidean metric u defined by $u(x, y) = \sqrt[2]{\sum_{i=1}^{n}(x_i - y_i)^2}$ for $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$ and $y = (y_1, y_2, \dots y_n) \in \mathbb{R}^n$.

2. C^n with the metric u defined by $u(x, y) = \sqrt[2]{\sum_{i=1}^{n} |x_i - y_i|^2}$ for $x = (x_1, x_2, \dots x_n) \in C^n$ and $y = (y_1, y_2, \dots y_n) \in C^n$.

3. Given any non empty set X, the metric d defined by $d(x, y) = 0$ if $x = y$ in X and $= 1$ if $x \neq y$ in X.

4. For $X = C[0,1]$, the metric ρ defined by $\rho(x, y) = \sup\{|x(t) - y(t)| : t \in [0,1]\}$ for all x, y in $C[0,1]$.

5. For X = l^{∞} = set of all bounded sequence of real or complex numbers, the metric d_∞ defined by $d_\infty(x, y) = \sup\{|x_n - y_n| : n \in N\}$ for all $x = (x_n)_n$, $y = (y_n)_n$ in l^{∞}.

6. Let $1 \leq p < \infty$. Consider the set l^p of all sequences $(x_n)_n$ of real or complex numbers such that $\sum_{n=1}^{\infty}|x_n|^p < \infty$. Define $d_p(x, y) = (\sum_{n=1}^{\infty}|x_n - y_n|^p)^{\frac{1}{p}}$ for all $x = (x_n)_n$, $y = (y_n)_n$ in l^p. Then d_p is a metric on l^p.

1.3. Property: If (x, d) is a metric space and x, y, z are in X then

$$|d(x,z) - d(y,z)| \leq d(x,y) \dots \dots \dots (1)$$

The proof follows from the Triangle inequality in the defining conditions of a metric and the symmetry in (1) between x and y.

1.4 Example of pseudometric space:

1. For $X = C[0,1]$, if ρ is defined by $\rho(x,y) = \inf\{|x(t) - y(t)| : t \in [0,1]\}$ for all x, y in $C[0,1]$ then ρ is a pseudometric on X.

1.5 Definition: Given a metric space X, if $a \in X, r > 0$ then $B_r(a) = \{x \in X : d(x,a) < r\}$ is called the open ball

and $B_r[a] = \{x \in X : d(x,a) \leq r\}$ is called closed ball with center a and radius r.

1.6 Definition: A subset S of a metric space X is said to be open in X if for every $a \in S$, there is some $r > 0$ such that $B_r(a)$ is a subset of S.

A subset F of X is said to be closed in X if $X \backslash F$ is open in X.

A point x in a metric space (X, d) is said to be an isolated point if {x} is an open set in X. (X, d) is said to be discrete if every point is isolated.

1.7 Examples: a) In any metric space (X, d), every open ball is an open set and every closed ball is a closed set.

b) If d be the discrete metric as defined in example 3 of **1.2** then every singleton subset is open as well as a closed set as for every $x \in X$, $\{x\} = B_{\frac{1}{2}}(x) = B_{\frac{1}{2}}[x]$.

c) The metric space (X, d) given in example 3 of **1.2** is a discrete metric space.

1.8 Theorem(Hausdorff property): For any two distinct points a, b in a metric space (X, d) there are two open sets U and V in X having the property:

$a \in U$, $b \in V$, $U \cap V = \emptyset$. In other words any two distinct points can be separated by open sets.

Proof: Since $a \neq b$ $d(a,b) > 0$. Let $r = \frac{d(a,b)}{2}$, $U = B_r(a)$and $V = B_r(b)$.

Then U and V serve the desire property.

The following properties can be directly established from the definitions:

1.9 Properties of open and closed sets: In any metric space (X, d)

a) Arbitrary union of open sets is again an open set.

b) Finite intersection of open sets is an open set.

c) $G \subseteq X$ is open in X if and only if G is the union of some open balls.

Using De Morgan's law on complementation of sets,

d) Arbitrary union of closed sets is again a closed set.

e) Finite intersection of closed sets is a closed set.

1.10 Definition: In a metric space (X, d) a subset $A \subseteq X$ is said to be a neighbourhood of a point a in X if there is a positive real number r such that $a \in B_r(a) \subseteq A$. In that case a is said to be an interior point of A. The set of interior points of a subset A of X is called the interior of A and is denoted by $\text{int}_X(A)$.

A point p is said to be an accumulation point of a subset A of the metric space (X, d) if for each $r > 0$, $B'_r(a) \cap A \neq \emptyset$. The set of all accumulation points of A is called derived set of A and it is denoted by A^d.

A point p is said to be an adherent point of a subset A of the metric space (X, d) if for each $r > 0$, $B_r(a) \cap A \neq \emptyset$. The set of all adherent points of A is called closure of A and it is denoted by $\text{cl}_X(A)$.

1.11 Remark: In a metric space (X, d) for each $a \in X$,

the countable family $N(a) = \{B_r(a) : r \text{ is a positive rational}\}$ forms a neighbourhood base at a. In that sense every metric space is a first countable topological space.

The following are straightforward from the definitions:

1.12 Properties: In any metric space (X, d)

a) $\text{int}_X(\emptyset) = \emptyset$, $\text{int}_X(X) = X$, $\text{cl}_X(\emptyset) = \emptyset$, $\text{cl}_X(X) = X$.

b) $G \subseteq X$ is open in X if and only if $G = \text{int}_X(G)$.

c) $\text{int}_X(G)$ is the largest open set in X contained in G.

d) $\text{int}_X(G) = \cup\{A \subseteq G : A \text{ is an open subset of } X\}$.

e) For any $A \subseteq X$, $\text{cl}_X(A) = A \cup A^d$.

f) For any $A \subseteq X$, $\text{cl}_X(A)$ is the smallest closed set in X containing A.

g) For any $A \subseteq X$, A is closed in X if and only if $A = \text{cl}_X(A)$.

h) For any $A \subseteq X$, A^d is a closed set in X.

i) For any $A \subseteq X$, $\text{cl}_X(X \setminus A) = X \setminus \text{int}_X(A)$ and $\text{int}_X(X \setminus A) = X \setminus \text{cl}_X(A)$.

j) For any $A \subseteq X$, $B \subseteq X$,

$$\text{(i) } A \subseteq B \text{ implies } \text{int}_X(A) \subseteq \text{int}_X(B), \text{cl}_X(A) \subseteq \text{cl}_X(B) \text{ (ii)} \text{int}_X(A \cap B) = \text{int}_X(A) \cap \text{int}_X(B), \quad \text{cl}_X(A \cup B) = \text{cl}_X(A) \cup \text{cl}_X(B).$$

1.13 Definition: A subset $A \subseteq X$ is said to be a $G_\delta -$ set ($F_\sigma -$ set) if it can be expressed as a countable intersection(union) of open(closed) subsets of (X, d).

1.14 Definition: A family B of open subsets of a metric space (X, d) is said to be a base for open sets if every open set can be expressed as a union of some members(possibly void) of B.

In a metric space (X, d) the family of open balls is a base for open sets.

1.15 Definition: In a metric space (X, d) a point a is said to be a boundary point of $A \subseteq X$ if a is neither an interior point of A nor an interior point of $X \setminus A$. The set of boundary points of A is called boundary of A and is denoted by $bd_X(A)$.

1.16 Distance between sets and diameter:

Let (X, d) be a metric space, $A \subseteq X$, $B \subseteq X$, $x \in X$. The distance between A and B denoted by d(A, B) is $d(A, B) = \inf\{d(p,q) : p \in A, q \in B\}$. The distance between A and the point x denoted by d(x, A) is $d(x, A) = \inf\{d(x,p) : p \in A\}$. The diameter of A denoted by diam(A) is diam(A) = $\sup\{d(p,q): p \in A, q \in A\}$.

The subset A is said to be bounded if diam(A) is finite. Otherwise it will be unbounded.

1.17 Properties: In a metric space (X, d) if $A \subseteq X, B \subseteq X$, $p \in X$, then

(i) $p \in cl_X(A)$ if and only if $d(p, A) = 0$.

(ii) $A \subseteq B \Rightarrow d(A) \leq d(B)$.

(iii) $d\big(cl_X(A), cl_X(B)\big) = d(A,B)$.

(iv) $d\big(cl_X(A)\big) = d(A)$.

(v) $d(A \cup B) \leq d(A) + d(B) + d(A,B)$.

1.18 Definition: Given a nonempty set X, two metric d and d_1 are said to be equivalent if every open set in (X, d) are open in (X, d_1) and vice versa.

1.19 Property: The following are equivalent for two metric spaces (X, d) and (X, m)
(i) Two metrics d and m on a set X are equivalent

(ii)There are two positive numbers r_1 and r_2 such that for all x, y in X,

$r_1 d(x,y) \leq m(x,y) \leq r_2\, d(x,y)$.

(iii There is a positive number c such that for all x, y in X,

$\frac{1}{c} d(x,y) \leq m(x,y) \leq c\, d(x,y)$.

1.20 Examples: (a) Given any metric space (X, d) the bounded metrics d_1 and d_2 defined by $d_1(x, y) = \frac{d(x, y)}{1+d(x,y)}$ for all x, y in X

and $d_2(x, y) = \min\{1, d(x, y)\}$ for all x, y in X are equivalent with d.

(b) On $\mathbb{R}^n$ the two metrics ρ^* and ρ^+ defined by, $\rho^*(x, y) = |x_1 - y_1| + |x_2 - y_2| + \ldots + |x_n - y_n|$

and $\rho^+(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|, \ldots, |x_n - y_n|\}$ are two equivalent metrics.

1.21 Metric subspace:

Suppose Y be a nonempty subset of a metric space (X,d). Then the restriction mapping of d on the set $Y \text{x} \, Y$ is indeed a metric on Y. The metric space (Y, d) is called a metric subspace of (X, d).

1.22 Remark: In a metric subspace (Y, d) as the metric d is the restriction of original metric d of (X, d) the open and closed balls in (Y,d) are precisely the intersection of Y with the open and closed balls of (X,d) respectively . Also the open and closed sets in (Y,d) are precisely the intersection of Y with the open and closed sets in (X, d) respectively.

1.23 Example: The metric subspace $\mathbb{N}$ of the Euclidean metric space $\mathbb{R}$ is a discrete metric space.

2. Completeness Property of Metric Spaces

One of the most important property of a first countable topological space is the its closed sets can be described by the convergence of the sequences from that set. We shall now discuss about the convergence of a sequence in a metric space.

2.1 Definition: In a metric space (X, d) a sequence $(x_n)_n$ is said to converge to a point $x \in X$ if for every $\varepsilon > 0$ there is a positive integer n_0 such that $n \geq n_0 \Rightarrow d(x_n, x) < \varepsilon$.

A sequence $(x_n)_n$ is said to be covergent if it converges to a point x in X .

A sequence $(x_n)_n$ is said to be a Cauchy sequence if for every $\varepsilon > 0$ there is a positive integer n_0 such that $n, m \geq n_0 \Rightarrow d(x_n, x_m) < \varepsilon$.

2.2 Note: If a sequence $(x_n)_n$ is convergent then it converges to a unique point(due to Hausdorff property of the metric space). The point is called the limit of the sequence $(x_n)_n$ and we denote it by $\lim_{n\to\infty} x_n$.

Also we often write $x_n \to x$ if $(x_n)_n$ converges to x .

2.3 Properties: In a metric space (X, d) if $A \subseteq X$, $x \in X, y \in X$ then

(i) If $(x_n)_n$ is convergent then it is a Cauchy sequence.

(ii) If $(x_n)_n$ is convergent then every subsequence of $(x_n)_n$ is convergent.

(iii) If $x_n \to x$ and $y_n \to y$ then $d(x_n, y_n) \to d(x, y)$.

(iv) $x \in cl_X(A)$ if and only if there is a sequence (x_n) in A such that $x_n \to x$.

(v) $x \in (A)^d$ if and only if there is a sequence (x_n) in $A \setminus \{x\}$ such that $x_n \to x$.

2.4 Theorem: In a metric space (X, d) if a Cauchy sequence $(x_n)_n$ has a convergent subsequence then the sequence $(x_n)_n$ is convergent.

Proof: Suppose $(x_n)_n$ be a Cauchy sequence which has a convergent subsequence $\left(x_{n_r}\right)_r$. Let $\left(x_{n_r}\right)_r$ converges to x. Then for each $\varepsilon > 0$, a positive integer k_1 such that $d(x_n, x_m) < \frac{\varepsilon}{2}$ for n, $m \geq k_1$. Also there is a positive integer k_2 such that $d\left(x_{n_r}, x\right) < \frac{\varepsilon}{2}$ for $r \geq k_2$. Choose $k = \max\,(n_{k_1}, k_2)$, then $n \geq k \Rightarrow$ $d(x_n, x) \leq d\left(x_n, x_{n_r}\right) + d\left(x_{n_r}, x\right) < \varepsilon$.

2.5 Theorem: In a metric space (X, d) every Cauchy sequence is bounded.

Proof: S*uppose* $(x_n)_n$ *be a Cauchy sequence. Choose a positive integer* k *such that* $d(x_n, x_m) < 1$ *for* $n,\ m \geq k$.

Let $b = 1 + max\{\ d(x_n, x_m): n, m \leq k\ \}$. *Then* $d(x_n, x_m) \leq b$ *for all* $n,\ m$.

The structure of the metric space will be very concrete in study of many interesting problems of mathematical analysis if we impose some extra property namely completeness.

2.6 Definition: A metric space (X, d) is said to be complete if every Cauchy sequence in (X, d) is convergent in (X, d).

A metric space (X, d) is said to be incomplete if it is not complete.

The completeness axiom for the real numbers is equivalent to the completeness of the metric space $\mathbb{R}$ and from this we conclude that $\mathbb{R}^n$ is also complete. Some nontrivial examples of complete metric spaces are:

2.7 Examples:

(a) The set $\mathrm{X} = \mathrm{C}[0,1]$ with the metric ρ defined by $\rho(\mathrm{x}, \mathrm{y}) = \sup\{|\mathrm{x}(\mathrm{t}) - \mathrm{y}(\mathrm{t})| : \mathrm{t} \in [0,1]\}$ for all x, y in C[0,1] is a complete metric space.

Proof: Let$(\mathrm{f_n})_\mathrm{n}$ be a Cauchy sequence in C[0, 1]. Then for every $\varepsilon > 0$

there is a positive integer $\mathrm{n_0}$ such that $\mathrm{n}, \mathrm{m} \geq \mathrm{n_0} \Rightarrow \rho(\mathrm{f_n}, \mathrm{f_m}) < \varepsilon$. This implies $\sup\{|\mathrm{f_n}(\mathrm{t}) - \mathrm{f_m}(\mathrm{t})| : \mathrm{t} \in [0,1]\} < \varepsilon$ for $\mathrm{n}, \mathrm{m} \geq \mathrm{n_0}$. Thus $|\mathrm{f_n}(\mathrm{t}) - \mathrm{f_m}(\mathrm{t})| < \varepsilon$ for every $\mathrm{t} \in [0,1]$ and for $\mathrm{n}, \mathrm{m} \geq \mathrm{n_0}$. So the sequence is uniformly convergent in [0, 1]. Since the uniform limit of a sequence of continuous functions is also a continuous function the sequence $(\mathrm{f_n})_\mathrm{n}$ is convergent. So C[0,1] is a complete metric space.

(b) If E is a measurable subset of $\mathbb{R}$ and $1 \leq \mathrm{p} \leq \infty$, $\mathrm{L^p} = \{\mathrm{f}: \mathrm{E} \rightarrow \mathbb{R} :$ f is measurable and $|\mathrm{f}|^\mathrm{p}$ is integrable over E$\}$ (with the assumption that $\mathrm{f} = \mathrm{g}$ in $\mathrm{L^p}$ iff $\mathrm{f} = \mathrm{g}$ a.e. on E) then ρ defined by $\rho(\mathrm{x}, \mathrm{y}) =$

$(\int |x-y|^p)^{\frac{1}{p}}$ for all x, y in L^p is a metric on L^p and from Riesz- Fischer theorem it can be shown that L^p is a complete metric space.

2.8 Some Examples of Incomplete metric spaces:

(a) The Euclidean metric space $\mathbb{Q}$ is an incomplete metric space.

(b) For $X = C[a, b]$, the metric ρ defined by $\rho(x, y) = \int_a^b |x(t) - y(t)|\, dt$ is an incomplete metric.

(c) Weierstrass's theorem tells us that the set P[a, b] of all real valued polynomial functions on [a, b] with respect to the supremum metric is an incomplete metric space.

In fact, the following is the characterisation of a subspace of a complete metric space to become a complete metric space.

2.9 Properties:

(1) Suppose (X, d) be a complete metric space and M be a nonempty subset of X. Then the metric subspace (M, d) is complete if and only if M is a closed subset of X.

Proof: If M is complete then any sequence $(x_n)_n$ in M which converges to a point $x \in X$, x can not be outside of M. So M is closed.

Conversely if M is a closed subset of (X, d) and if a sequence $(x_n)_n$ is Cauchy in M then it is Cauchy in (X, d) also. X being complete, $(x_n)_n$ converges to a point x in X. Then x is an adherent point of M. M being closed, x must be in M. So M is complete.

(2) If (X, d) and (Y, ρ) are two metric spaces, then the product space X x Y is complete if and only if both X and Y are complete metric spaces.

Proof: The result follows from the fact that two sequences $(x_n)_n$ in X and $(y_n)_n$ in Y are Cauchy (convergent)sequences in the respective metric space if and only if $(x_n, y_n)_n$ is Cauchy (convergent)sequence in X x Y.

(3) (Cantor's intersection theorem): Let $(F_n)_n$ be a contracting sequence of nonempty closed sets in a metric space (X, d) such that $\text{diam}(F_n) \to 0$ as $n \to \infty$. Then (X, d) is complete if and only if $\bigcap_n F_n$ is a singleton set.

Proof: First assume that X is complete. Let $(F_n)_n$ be a contracting sequence of nonempty closed sets in a metric space (X, d) such that $\text{diam}(F_n) \to 0$ as $n \to \infty$. For each index n select $x_n \in F_n$. We claim that $(x_n)_n$ is a Cauchy sequence. Indeed, for each $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that $\text{diam}(F_{n_0}) < \varepsilon$. Since $(F_n)_n$ is contracting if $n, m \geq n_0$ then $x_n, x_m \in F_{n_0}$ and $d(x_n, x_m) \leq \text{diam}(F_{n_0}) < \varepsilon$. Since X is complete $(x_n)_n$ converges to some $x \in X$. However for each index n, F_n is closed and $x_k \in F_n$ for $k \geq n$ so that $x \in F_n$ for all n. So $\bigcap_n F_n \neq \emptyset$. If another y distinct from x is in $\bigcap_n F_n$, then $\text{diam}(F_n) \geq d(x, y) \nrightarrow 0$ leads to a contradiction.

To prove the converse, suppose that for any contracting sequence $(F_n)_n$ of nonempty closed subsets of X, there is a point $x \in X$ for which $\bigcap_n F_n = \{x\}$. Let $(x_n)_n$ be a Cauchy sequence in (X, d). For each index n, let F_n be the closure of the nonempty set $\{x_k : k \geq n\}$. Then $(F_n)_n$ is a contracting sequence of nonempty closed sets . So $\bigcap_n F_n = \{x\}$ for some $x \in X$. Clearly then $(x_n)_n$ converges to x. Therefore X is complete.

Roughly speaking a metric space fails to be complete because it has "holes". If X is an incomplete metric space, it can always be suitably minimally enlarged to become complete.

2.10 Definition: Given two metric spaces (X, d) and (Y, ρ) a mapping $f: X \to Y$ is said to be an isometry from X into Y if $d(p, q) = \rho(f(p), f(q))$ for every pair $p, q \in X$. X and Y are said to be isometric if there is an isometry from X onto Y.

A completion of a metric space (X, d) is a complete metric space (Y, ρ) such that there is an isometry f from X into Y and f(X)is dense in Y.

The following theorem ensures that there is a completion of a metric space which is unique in sense of isometry.

2.11 Theorem: Every metric space has a completion. Also if $(\overline{X}, \rho)$ and $(\overline{X}_1, \rho_1)$ are two completion of a metric space (X, d) then they are isometric.

Proof: Suppose (X, d) be a metric space.

Step 1. Let X' denote the set of Cauchy's sequences on X. Let us define a relation $\sim$ on X' by $(x_n)_n \sim (y_n)_n$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$. Then $\sim$ is an equivalence relation on X'. Let $\overline{X} = X'/\sim$ be the set of equivalence classes.

Step 2. Let us define $\rho : \overline{X} \times \overline{X} \to \mathbb{R}$ by

for all $(x_n)_n$, $(y_n)_n$ in $\overline{X}$, $\rho((x_n)_n, (y_n)_n) = \lim_{n\to\infty} d(x_n, y_n)$.

The mapping is well defined. It is a routine matter to check that ρ is a metric on $\overline{X}$.

Step 3. We shall now show that $\overline{X}$ is a complete metric space.

Let $(x_n^m)_{n,m}$ be a Cauchy sequence in $\overline{X}$. Then $\rho\left(\left(x_n^p\right)_n, \left(x_n^q\right)_n\right) \to 0$ as $p, q \to \infty$.

Also for each m, $(x_n^m)_{n,m}$ is a Cauchy sequence in (X, d). So we may assume that $d(x_{n+1}^m, x_n^m) < \frac{1}{2^n}$ for each n. Let $y_n = x_n^n$. Then $y = (y_n)_n$ is a Cauchy sequence in (X, d) indeed $d\left(y_p, y_q\right) = d\left(x_p^p,\ x_q^q\right) \to 0$ as $p, q \to \infty$. Also $\rho((x_n^m)_n,\ y) \to 0$ as $m \to \infty$ implies $(x_n^m)_{n,m}$ is convergent.

Step 4. Suppose f: X → $\overline{X}$ be the mapping defined by $f(x) = (x)_n$, the constant sequence.

Then for x, y in X, $\rho\left(f(x),\ f(y)\right) = \lim_{n\to\infty} d(x,\ y) = d(x,\ y)$ implies f is an isometry.

Also for each $\bar{x} = (x_n)_n$ in $\overline{X}$, and for each $\varepsilon > 0$, there is $k \in \mathbb{N}$ such that

$d(x_n,\ x_m) < \varepsilon$ for n, $m \geq k$. Let $x = x_k$. Then $f(x) = (x) \in B_\varepsilon(\bar{x})$ implies $f(X)$ is dense in $\overline{X}$.

Consequently $\overline{X}$ is a completion of X.

Step 5. Let $(\overline{X}_1,\ \rho_1)$ be another completion of X where $g: X \to \overline{X}_1$ be an isometry with $g(X)$ dense in $\overline{X_1}$. We have to define a mapping π from $\overline{X}$ onto $\overline{X}_1$ which will be an isometry. Pick any $\bar{x}$ from $\overline{X}$. There is a sequence $\left(f(x_n)\right)_n$ in $f(X)$ converges to $\bar{x}$. $f(x_n)$ being a constant sequence, $\rho(x_n, \bar{x}) \to 0$ as $n \to \infty$.

This implies $(x_n)_n$ is a Cauchy sequence in (X, d). Then $\big(g(x_n)\big)_n$ is a Cauchy sequence in $(\overline{X}_1,\ \rho_1)$. $\overline{X}_1$ being complete, $\big(g(x_n)\big)_n$ converges to some $\overline{x_1}$. Denote $\overline{x_1}$ by $\pi(\overline{x}\,)$.

Then π is well defined. It can be easily shown that π is an isometry

between $\overline{X}$ and $\overline{X}_1$.

Baire's Category Theorem

The concept of first and second category are ways of describing in a certain sense, the size of a metric space. They are based in turn on the concept of a nowhere dense set.

2.12 Definition: A subset A of a metric space (X, d) is said to be nowhere dense set in X if $\mathrm{int}_X\big(\mathrm{cl}_X(A)\big) = \emptyset$. In other words the closure of A does not contain a nonempty open set.

2.13 Examples: (i) Every finite subset of the Euclidean metric space $\mathbb{R}$ is a nowhere dense set in $\mathbb{R}$.

(ii) In a metric space (X, d) any set consisting of a convergent sequence (with or without limit) is a nowhere dense set.

(iii) The set of rational numbers is not a nowhere dense set in the set $\mathbb{R}$ with Euclidean metric.

(iv) The set of irrational numbers is also not a nowhere dense set in the set $\mathbb{R}$ with Euclidean metric.

(v) The Cantor's set (constructing by removing middle 1/3 open interval in each step from [0, 1]) is an uncountable subset of [0, 1] which is a nowhere dense set.

2.14 Properties : In a metric space (M, d) (i) an open subset A is dense in (M, d) if and only if M \ A is nowhere dense in (M, d).(ii) If M has no isolated point then closure of a discrete set is a nowhere dense set. (iii) the boundary of an open set is closed and nowhere dense. (iv) every closed nowhere dense set is the boundary of an open set.

Proof: (i) M $\setminus$ A is nowhere dense in (M, d) $\Leftrightarrow \mathrm{int_M}(\ \mathrm{cl_M}(\ \mathrm{M}\ \setminus \mathrm{A})) = \emptyset \ \ \Leftrightarrow \mathrm{M}\ \setminus \mathrm{cl_M}(\mathrm{A}) = \ \emptyset \ \Leftrightarrow \ \left(\mathrm{cl_M}(\mathrm{A})\right) = \mathrm{M} \Leftrightarrow$ A is dense in M.

(ii) Let D be a discrete subset of M. Suppose D is not a nowhere dense set. Then we have a nonempty open set $\mathrm{U} \ \subseteq \mathrm{cl_M}(\mathrm{D})$. There must be an element a $\in \mathrm{U} \cap \mathrm{D}$. Then there exists an $\mathrm{r} > 0$ such that $\mathrm{B_r}(\mathrm{a}) \ \subseteq \mathrm{U}$ and contains no other points of D. Since M has no isolated points, there is a $\mathrm{y} \neq \mathrm{a}$ in $\mathrm{B_r}(\mathrm{a})$. Then every open ball centred y must intersect D, contradicts that D is discrete.

(iii) Let U be an open set in (M, d). Then $\mathrm{bd_M}(\mathrm{U}) = \mathrm{cl_M}(\mathrm{U}) \ \setminus \mathrm{int_M}(\mathrm{U}) = \mathrm{cl_M}(\mathrm{U}) \ \setminus \mathrm{U}$ is a closed set. Also if A is a nonempty open set contained in $\mathrm{bd_M}(\mathrm{U})$ then there is some $\mathrm{y} \ \in \mathrm{U^d} \cap \mathrm{A}$. Since A is open there is some $\mathrm{r} > 0$ such that $\mathrm{B_r}(\mathrm{y}) \subseteq \mathrm{A} \ \subseteq \ \mathrm{cl_M}(\mathrm{U}) \ \setminus \mathrm{U}$ a contradiction. So $\mathrm{bd_M}(\mathrm{U})$ is nowhere dense set.

(iv) Suppose F be a closed nowhere dense subset of M. Let $\mathrm{U} = \mathrm{M} \ \setminus \mathrm{F}$. Then U is a dense open set and $\mathrm{F} = \mathrm{M} \ \setminus \mathrm{U} = \mathrm{cl_M}(\mathrm{U}) \setminus \mathrm{U} = \mathrm{bd_M}(\mathrm{U})$.

2.15 Definition: A subset of a metric space is of (i) the first category if it is expressible as a countable union of nowhere dense sets.

(ii) the second category if it is not of the first category.

We shall now state the theorem which is known as Baire’s category theorem.

2.16 Theorem: Every complete metric space is of second category.

Proof: Suppose on the contrary that the complete metric space (M, d) is a countable union $\bigcup_n \mathrm{A_n}$ of nowhere dense sets. We begin a construction which leads us to a contradiction. $\mathrm{A_1}$ being nowhere dense, it will be disjoint from a ball. We can take it to be a closed ball $\mathrm{S_1}$ of radius ≤ 1. $\mathrm{A_2}$ being nowhere dense, it will be disjoint from a ball. So there is a closed ball $\mathrm{S_2}$ of radius $\leq \frac{1}{2}$ and $\mathrm{S_2} \ \subseteq \mathrm{S_1}$ and $\mathrm{S_2} \cap \mathrm{A_2} = \emptyset$. Continuing in this way we get a descending sequence $(\mathrm{S_n})_n$ of nonempty closed balls with $\mathrm{S_n} \cap \mathrm{A_n} = \emptyset$ and radius($\mathrm{S_n}$) $\leq \frac{1}{n}$. By Cantor’s intersection theorem there is a point y in $\bigcap_n \mathrm{S_n}$. But y lies in none of $\mathrm{A_n}$’s, contradicting the hypothesis that $\mathrm{M} = \ \bigcup_n \mathrm{A_n}$.

2.17 Note: Completeness property is a metric property i.e. preserved under any isometry, whereas second category property is a topological property. So the theorem is a link between metric property and topological property.

2.18 Examples: (a) The set of rational numbers is of first category.

(b) Denumerable union of first category subsets of a metric space is also of first category.

(c) From Baire's category theorem it can be deduce that the set of irrational numbers is a set of second category.

3. Continuity on Metric Spaces

3.1 Definition: A function f: (X,d) → (Y, ρ) is said to be continuous at a point a $\in X$ if for every positive ε there is a positive ∂ such that d(x, a) < ∂ implies ρ(f(x), f(a)) < ε. A function f is continuous on X, if it is continuous at every point of X.
Since a metric space is first countable the continuity of a function at a point can be characterised by sequential criteria:
3.2 Theorem: A function f: (X,d) → (Y, ρ) is continuous at a point a $\in X$ if and only if
$\left(f(x_n)\right)_n$ converges to $f(a)$ in (Y,ρ) when every $(x_n)_n$ converges to a in (X,d).
Proof: If f is continuous at a point a $\in X$ and $(x_n)_n$ converges to a in (X,d) then for each $\varepsilon > 0$ there is a $\partial > 0$ such that d(x, a) < ∂ implies ρ(f(x), f(a)) < ε. Then there is a positive integer k such that $d(x_n, a) < \partial$ for $n \geq k$. Then $\rho(f(x_n), f(a)) < \varepsilon$ for $n \geq k$ implies $\left(f(x_n)\right)_n$ converges to $f(a)$ in (Y,ρ).
Conversely if possible let f be not continuous at a. Then there is some $\varepsilon > 0$ such that for every choice of $\partial > 0$ there is $x_{\frac{1}{\partial}}$ in X such that d($x_{\frac{1}{\partial}}$, a) < ∂ but $\rho\left(f(x), f(a)\right) \geq \varepsilon$. Taking $\partial = \frac{1}{n}$ for each n, there is a sequence $(x_n)_n$ in X, converging to a but $\left(f(x_n)\right)_n$ does not converge to $f(a)$ in (Y,ρ) leads to a contradiction.
3.3 Definition: A function f: (X,d) → (Y, ρ) is said to be uniformly continuous on X if for every positive ε there is a positive ∂ such that d(x, y) < ∂ implies ρ(f(x), f(y)) < ε for any two x, y in X.

From the definition it is clear that every uniformly continuous function is continuous.

3.4 Theorem: A function f: (X,d) → (Y, ρ) is uniformly continuous on X if and only if $\left(f(x_n)\right)_n$ is a Cauchy sequence in (Y,ρ) when ever $(x_n)_n$ is a Cauchy sequence.
Proof: Similar to the proof of theorem 3.2
3.5 Theorem: If function f: (X,d) → (Y, ρ) is continuous at a point a $\in X$ and function g: (Y, ρ) → (Z, σ) is continuous at a point f(a) $\in Y$ then their composition gof: (X,d) → (Z, σ) is continuous at a.
Proof: Follows directly from the theorem 3.2

3.6 Theorem: for a function $f:(X,d) \to (Y,\rho)$ the following are equivalent:

(a) f is continuous on X .

(b) $f^{-1}(G)$ is open in (X,d) for every open set G in (Y,ρ).

(c) $f^{-1}(K)$ is closed in (X,d) for every closed set G in (Y,ρ).

Proof: (a) $\Rightarrow$ (c): If $p \in cl_X(f^{-1}(K))$ then there is a sequence $(x_n)_n$ in $f^{-1}(K)$converges to p in (X,d). From the continuity of f at p, $(f(x_n))_n$ converges to f(p) in (Y,ρ). K being closed $f(p) \in K$. So $p \in (f^{-1}(K))$ implies (c)

(c) $\Rightarrow$ (b): For every open set G in (Y,ρ) $Y \setminus G$ is closed. Then (c) implies $f^{-1}(Y \setminus G)$ is closed in (X,d) which implies $X \setminus f^{-1}(G)$ is closed in X. So (b) holds.

(b) $\Rightarrow$ (a): For every positive ε, $B_\varepsilon(f(a))$ is an open set containing f(a). Then $f^{-1}(B_\varepsilon(f(a)))$ is open set containing a in (X,d). Choose positive ∂ such that $B_\partial(a) \subseteq f^{-1}(B_\varepsilon(f(a))$. So $f(B_\partial(a)) \subseteq B_\varepsilon(f(a)$ which implies f is continuous at a.

3.7 Theorem: For any subset A of a metric space (X,d) the mapping $f: X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ for $x \in X$ is a uniformly continuous mapping.

Proof: The result follows immediately from the fact that for any $x, y \in X$, $|f(x) - f(y)| = |d(x, A) - d(y, A)| \leq d(x,y)$.

3.8 Theorem: For any closed subset A of a metric space (X,d) there is a continuous mapping $f: X \to \mathbb{R}$ such that $f(x) = 0$ if and only if $x \in A$.

Proof: Define $f: X \to \mathbb{R}$ by $f(x) = d(x, A)$ for $x \in X$. From the previous theorem 3.7 f is a continuous mapping on X and $f(x) = 0$ if and only if $d(x,A) = 0$ if and only if $x \in cl_X(A) = A$.

3.9 Remark: From theorem 3.6 and 3.8 it is clear that in a metric space a subset is closed if and only if it is a zero set.

3.10 Regularity Property: For any closed subset A of a metric space $(X,d)$and $x \in X \setminus A$ there are two open sets U and V in (X, d)such that $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$

Proof: Define $f: X \to \mathbb{R}$ by $f(z) = d(z, A)$ for $z \in X$. Then $d(x, A) = 2r > 0$ From 3.7 f is a continuous mapping on X. Let $U = f^{-1}(-1, r)$ and $V = f^{-1}(r, 3r)$. Then both U and V are open sets. Then $A \subseteq U$, $x \in V$ and $U \cap V = \emptyset$

3.11 Property: Suppose A and B are two disjoint closed subsets of a metric space (X, d). Then there are two open sets U and V in (X, d) such that $A \subseteq U$, $B \subseteq V$ and $U \cap V = \emptyset$ (This property is called normal property).

Proof: Define $f: X \to \mathbb{R}$ by $f(z) = \frac{d(z,A)}{d(z,A)+d(z,B)}$ for $z \in X$. A and B being disjoint f is well defined. Also from theorem 3.7 and 3.8 f is continuous function assumes values 0 on A and 1 on B. Considering $U = f^{-1}\left(-1, \frac{1}{2}\right)$ and $V = f^{-1}(\frac{1}{2}, 2)$ we obtain the desire result.

3.12 Property: For any metric space (X, d) the function $d: X \times X \to \mathbb{R}$ is a continuous mapping.

Proof: For any $(x_0, y_0) \in X \times X$, if a sequence $(x_n, y_n)_n$ converges to (x_0, y_0) then from 2.3 (iii) $\left(d(x_n, y_n)\right)_n$ converges to $d(x_0, y_0)$. Hence the function $d: X \times X \to \mathbb{R}$ is a continuous mapping.

3.13 Theorem: for a continuous function f: (X,d) → (Y, ρ) the graph of f $\{(x, f(x)): x \in X\}$ is a closed subset of $X \times Y$.

Proof: Given any convergent sequence $\left(x_n, f(x_n)\right)_n$ in $X \times Y$ if $\lim_{n\to\infty} (x_n, f(x_n)) = (a, b)$ then $x_n \to a$ in (X, d) and $f(x_n) \to b$ in (Y, d). Continuity of f at a implies $b = f(a)$. So the graph of f is closed.

3.14 Theorem: If two continuous functions f, g: (X,d) → (Y, ρ) agree in a dense subset of X, then they agree in the whole space X.

Proof: The result follows immediately from the sequential criteria.

3.15 Theorem: Suppose A be a dense subset of a metric space (X, d) and (Y, ρ) be a complete metric space. Then every uniformly continuous function $f: A \to Y$ can be uniquely extended to a uniformly continuous function $g: X \to Y$.

Proof: We shall draw the sketch of the proof in the following way: since A is dense in X for any $x \in X$, choose a sequence $(a_n)_n$ from A converging to x. Then $\left(f(a_n)\right)_n$ will be a Cauchy sequence in the complete metric space (Y, ρ).

Define $g(x) = \lim_{n} f(a_n)$. Then g is well defined. Clearly g extends f. Uniform continuity of g can be easily established. Uniqueness follows from theorem 3.13.

3.16 Definition: For two metric spaces (X, d) and (Y, ρ) a mapping f: $(X, d) \to (Y, \rho)$

is said to be a homeomorphism if f is bijective , f and f^{-1} are continuous.

3.17 Definition : (X, d) and (Y, ρ) be metric spaces and $f: X \to Y$ be a function. For a point $a \in X$, and for any positive ∂, let us define $\Omega(f, B_\partial(a)) = \sup\{\rho(f(x), f(y)): x \in B_\partial(a), y \in B_\partial(a)\}$. $\Omega(f, B_\partial(a))$ is called oscillation of f over $B_\partial(a)$.

3.18 REMARKS: (1) If f is unbouned in any deleted neighbourhood of the point a, then $\Omega\left(f, B_\partial(a)\right)$ will be infinity. So for a bounded function f, $\Omega(f, B_\partial(a))$ must be finite for every ∂.

(2) If f is a bounded function then the oscillation function $\Omega(f, B_\partial(a))$ is increasing function. So $\lim_{\partial\to 0} \Omega(f, B_\partial(a))$ exists and $\lim_{\to 0} \Omega(f, B_\partial(a)) = \inf\{\Omega(f, B_\partial(a)) : \partial > 0\}$.

(3) If $B_{\partial_1}(b) \subset B_\partial(a)$ then $\Omega\left(f, B_{\partial_1}(b)\right) \leq \Omega\left(f, B_\partial(a)\right)$.

3.19 Definition: If f is a bounded function from a metric space X to R, and a $\in$ X, then the oscillation of the function f at the point a is defined by $\lim_{\partial\to 0} \Omega(f, B_\partial(a))$ and it is denoted by $\omega(f, a)$.

3.20 Theorem: Suppose f: (X,d) → (Y, ρ) be a function and a $\in X$. Then the necessary and sufficient condition for the continuity of f at a is $\omega(f, a) = 0$.

Proof: Suppose f is continuous at a. For any positive ε there is a positive ∂ such that d(x, a) < ∂ implies ρ(f(x), f(a)) < $\frac{\varepsilon}{2}$. Then for any p, q $\in B_\partial(a)$, $\rho\left(f(p), f(q)\right) \leq \rho\left(f(p), f(a)\right) + \rho\left(f(a), f(q)\right) < \epsilon$.

So $\Omega\left(f, B_\partial(a)\right) \leq \epsilon$ and hence $\omega(f, a) = 0$.

Conversely, suppose $\omega(f, a) = 0$. Now for any $\epsilon > 0$, there is some positive ∂ such that $\Omega\left(f, B_\partial(a)\right) < \epsilon$.

By definition, ρ(f(x), f(a)) < ε whenever d(x, a) < ∂. Which shows that f is continuous at the point a.

3.21 Notation: For a function f: (X,d) → (Y, ρ), the set $\{x \in X: \text{f is not continuous at the point x}\}$ is denoted by D_f.

3.22 Properties: Suppose f: (X,d) → (Y, ρ) be a function. Then

(i)The set $D_n = \{x \in X : \ \omega(f, x) \geq \frac{1}{n}\}$ is a closed subset of X.
(ii) The set D_f is an F_σ set.
Proof: (i) If $x_0 \notin D_n$ then $\omega(f, x_0) < \frac{1}{n}$. That is equivalent to $\inf\{\Omega(f, B_\partial(a)) : \partial > 0\} < \frac{1}{n}$. So for any $\epsilon > 0$, there is some positive ∂ such that $\Omega(f, B_\partial(x_0)) < \frac{1}{n}$.
Then for any $x \in B_\partial(x_0)$, choose $\partial_1 > 0$ such that $B_{\partial_1}(x) \subset B_\partial(x_0)$.
Then from the earlier remark $\Omega\left(f, B_{\partial_1}(x)\right) \leq \Omega(f, B_\partial(x_0)) < \frac{1}{n}$ implies $x \notin D_n$.
Thus $B_\partial(x_0) \cap D_n = \phi$ which shows that D_n is a closed set.
(ii) Observe that $D_f = \bigcup_{n=1}^{\infty} D_n$ is the countable union of closed sets.

We want to now focus on our main problem: whether there be a real valued function defined on R which is continuous at all rational points and discontinuous at all irrational points. To make the conclusion we will apply Baire's Category theorem.

3.23 Proposition: There does not exit a function $f : [0, 1] \rightarrow \mathbb{R}$ which is discontinuous only at all irrational points in $[0, 1]$.
Proof: We shall prove this by using contradiction. If possible let there be a function $f : [0, 1] \rightarrow \mathbb{R}$
such that $D_f = [0, 1] \setminus Q$. From 1.6(ii) D_f is an F_σ set. So $[0, 1] \setminus Q = \bigcup_{n=1}^{\infty} F_n$ where each F_n is a closed set. As $F_n \subset [0, 1] \setminus Q$, it contains no interior point. So $\text{int}_X(\text{cl}_X(F_n)) = \phi$ where $X = [0, 1]$ with the usual metric. Consequently each F_n is a nowhere dense set. So $[0, 1] \setminus Q$ is a set of first category.

Also $[0, 1] \cap Q$ is a set of first category. So $[0, 1] = ([0, 1] \setminus Q) \cup ([0, 1] \cap Q)$ is of first category contradicts the Baire's category theorem.

Therefore no such f can be found.

4 Compact Metric Space and Totally Boundedness

4.1 Definition: A family $(E_\gamma)_{\gamma \in \Gamma}$ of open sets in a metric space (X, d) is said to be an open cover of a subset $E \subseteq X$ if $E \subseteq \bigcup_{\gamma \in \Gamma} E_\gamma$. A sub collection of $(E_\gamma)_{\gamma \in \Gamma}$ is called a sub cover of $(E_\gamma)_{\gamma \in \Gamma}$ if it is also an open cover of E.

A sub set $E \subseteq X$ is said to be (a) compact if every open cover of E has a finite sub cover. (b) lindelof if every open cover of E has a countable sub cover. (c)$\sigma -$ compact if every countable open cover of E has a finite sub cover.

4.2 Proposition: For a metric space (X, d) the following are equivalent:

(a) X is compact.

(b) If $(F_\gamma)_{\gamma\in\Gamma}$ be a family of closed subsets of X such that intersection of any finite subfamily is nonempty then $\bigcap_{\gamma\in\Gamma} F_\gamma \neq \emptyset$.

Proof: (a) $\Rightarrow$ (b): If $\bigcap_{\gamma\in\Gamma} F_\gamma = \emptyset$ then $\bigcup_{\gamma\in\Gamma}(X \setminus F_\gamma) = X$, where each $X \setminus F_\gamma$ is an open set X. So the family $(X \setminus F_\gamma)_{\gamma\in\Gamma}$ is an open cover of X. Since X is compact, it has a finite subcover, say there is a finite subset $\Gamma_1 \subseteq \Gamma$ such that $\bigcup_{\gamma\in\Gamma_1}(X \setminus F_\gamma) = X$, which implies $\bigcap_{\gamma\in\Gamma_1} F_\gamma = \emptyset$ leads to a contradiction.

(b) $\Rightarrow$ (a): Suppose $(G_\gamma)_{\gamma\in\Gamma}$ be an open cover of X. Then $\bigcup_{\gamma\in\Gamma} G_\gamma = X$ implies $\bigcap_{\gamma\in\Gamma}(X \setminus G_\gamma) = \emptyset$, where $X \setminus G_\gamma$ is a closed set. So there is a finite subset $\Gamma_1 \subseteq \Gamma$ such that $\bigcap_{\gamma\in\Gamma_1}(X \setminus G_\gamma) = \emptyset$ which again implies $\bigcup_{\gamma\in\Gamma_1} G_\gamma = X$. Thus each open cover has a finite subcover and so X is compact.

4.3 Theorem: Every closed subset of a compact metric space is compact.

Proof: Let Y be a closed subset of a compact metric space (X, d). If $(F_\gamma)_{\gamma\in\Gamma}$ be a family of closed subsets of Y such that intersection of any finite subfamily is nonempty then $F_\gamma = Y \cap K_\gamma$ for some closed subset K_γ of X. So each F_γ is closed in (X, d). Since (X, d) is compact $\bigcap_{\gamma\in\Gamma} F_\gamma \neq \emptyset$. Hence Y is compact.

4.4 Theorem: Suppose f: $(X,d) \to (Y, \rho)$ be a continuous function and X is a compact metric space. Then f is uniformly continuous.

Proof: Choose $\varepsilon > 0$ arbitrarily. Now for each $x \in X$ using continuity of f, there is a $\delta_x > 0$ such that for any $y \in X$ with $d(x, y) < \delta_x$ implies $\rho(f(x), f(y)) < \frac{\varepsilon}{2}$. Now the family $\left\{ B_{\frac{\delta_x}{2}}(x) : x \in X \right\}$ being an open cover of the compact set X there are finite number of points, say $x_1, x_2, \dots x_n$ in X such that $\left\{ B_{\frac{\delta_{x_i}}{2}}(x_i) : i = 1,2, \dots n \right\}$ also covers X. Let $\delta = \frac{1}{2}\min\{\delta_{x_i} : i = 1,2, \dots n\}$. Let $u, v \in X,$ and $d(u, v) < \delta$. If $u \in B_{\frac{\delta_{x_k}}{2}}(x_k)$ then $d(x_k, v) \leq d(x_k, u) +$

$d(u,\ v) < \frac{\delta_{x_k}}{2} + \frac{\delta_{x_k}}{2} = \ \delta_{x_k}$. Then $\rho\big(f(u), f(v)\big) \leq \ \rho\big(f(u), f(x_k)\big) + \rho\big(f(x_k), f(v)\big) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ showing that f is uniformly continuous.

4.5 Theorem: Suppose f: (X,d) → (Y, ρ) be a continuous function and X is a compact metric space. Then f(X) is a compact subset of Y.

Proof: Suppose $\left(G_\gamma\right)_{\gamma\in\Gamma}$ be an open cover of $f(X)$. Since f is continuous $f^{-1}(G_\gamma)$ is an open set in X. So $\left(f^{-1}(G_\gamma)\right)_{\gamma\in\Gamma}$is an open covering of X. X being compact, $\left(f^{-1}(G_\gamma)\right)_{\gamma\in\Gamma_1} = X$ for some finite subset Γ_1 of Γ. This implies $\left(G_\gamma\right)_{\gamma\in\Gamma_1}$ is a finite sub cover of $\left(G_\gamma\right)_{\gamma\in\Gamma}$.Hence f(X) is a compact subset of Y.

4.6 Theorem: Suppose f: (X,d) → (Y, ρ) be a continuous bijective function and X is a compact metric space. Then f is a homeomorphism.

Proof: We have only to show that f^{-1} is continuous. Suppose K be a closed subset in (X, d). Then from 4.3 K is a compact subset of (X, d) which implies $f(K)$ is a compact set in (Y, ρ). Hence $f(K)$ is closed in (Y, ρ). So f^{-1} is continuous.

4.7 Definition: A metric space X is said to be totally bounded if for every $\epsilon > 0$, there is a finite family of open balls of radius ϵ which covers X . A subset Y of X is called totally bounded provided that Y as a metric subspace of X, is totally bounded.

4.8 Remark: For a subset Y of a metric space X, by an $\epsilon - \text{net}$ for Y we mean a finite family of open balls $\left(B_\epsilon(x_n)\right)_{n=1}^{k}$ with center $x_n \in X$ which covers Y. Consequently a metric subspace Y is totally bounded if and only if there is a finite $\epsilon - \text{net}$ for Y. Also from the existence of finite $\epsilon - \text{net}$ it is clear that the diameter of a totally bounded metric space is finite and so every totally bounded metric space is bounded.

4.9 Example: Consider the metric space $X = \ l^2$ of all sequences in $\mathbb{C}$ which are square summable.

Here the closed unit ball **B** = $\{x = (x_n)_n:\ \ (\sum_{n=1}^{\infty} |x_n|^2\ )^{\frac{1}{2}} \leq\ 1\}$ is bounded. But for each $n \in \mathbb{N}$, if $e_n = (\delta_k^n)_k$ (where δ_k^n is the Kronecker delta) then $\rho(e_n, e_m) = \ \sqrt{2}$ for $m\ \neq n$. Thus **B** can not be covered by a finite number of balls radius less than ½ . So **B** is not totally bounded.

In the Euclidean metric space the following proposition tells something different:

4.10 Proposition: A subset of Euclidean space $\mathbb{R}^n$ is totally bounded if and only if it is bounded.

4.11 Definition: A metric space (X, d) is said to be sequentially compact if every sequence in X has a subsequence which converges to a point of X.

4.12 Theorem: A metric space (X, d) is totally bounded if and only if every sequence in X contains a Cauchy subsequence.

4.13 Theorem: In a metric space (X, d) the following are equivalent:

(a) X is complete and totally bounded.

(b) X is compact. (c) X is sequentially compact.

Proof: (a) implies (b): Suppose on the contrary that $(G_\gamma)_{\gamma\in\Gamma}$ be an open cover of X which has no finite sub cover. Since X is totally bounded there is a finite family of open balls of radius < ½ that cover X. At least one of that family can't be covered by a finite subfamily of $(G_\gamma)_{\gamma\in\Gamma}$. Let us denote the closure of this ball by F_1. Again using totally boundedness of X there is a finite family of open balls of radius < ¼ that cover X(and hence coverF_1) . At least one of that family whose intersection with F_1 can't be covered by a finite subfamily of $(G_\gamma)_{\gamma\in\Gamma}$. Let us denote the closure of this ball by F_2. Continuing this process we obtain a descending sequence of nonempty closed sets $(F_n)_n$ with $\operatorname{diam}(F_n) \to 0$. Since X is complete from Cantor's intersection theorem, $\bigcap_n F_n$ is a singleton set which contradicts that no F_n can be covered by finite subfamily of $(G_\gamma)_{\gamma\in\Gamma}$. So (X, d) is compact.

(b) implies (c): If $(x_n)_n$be any sequence in (X, d) then for every index n, let $F_n = \mathrm{cl}_X\{x_k : k \geq n\}$. Then $(F_n)_n$ is a descending sequence of nonempty closed sets and so it has finite intersection property. By compactness of X $\bigcap_n F_n$ contains a point, say $x_0 \in X$. Clearly there is a subsequence of $(x_n)_n$ which converges to x_0.

(c) implies (a): Since X is sequentially compact, every sequence in X has a convergent and hence a Cauchy subsequence. From theorem 4.12 it follows that X is totally bounded. To show that X is complete, let $(x_n)_n$ be a

Cauchy sequence. From sequential compactness $(x_n)_n$ has a convergent subsequence and so $(x_n)_n$ itself is convergent.

5 Fixed Point Theorems and Their Applications.

5.1 Definition: A point x in a set X is said to be a fixed point of the mapping $T: X \to X$ if $T(x) = x$.

The fixed point of a real valued function of a real variable x corresponds to a point in the plane where the graph of the function intersects with the line $y = x$. Analytically, using intermediate value property, also we can ensure the existence of a fixed point of a continuous map from $[a, b]$ to $[a, b]$. Brouwer's fixed point theorem ensures that any continuous map from a compact convex subset of R^n to itself has a fixed point . In this section our aim is to impose certain conditions on the mapping to ensure the fixed point in a metric space.

5.2 Definition: Given a metric space (X, d) a mapping $T: X \to X$ is said to be (i) Lipshcitz if $d(T(x), T(y)) \leq c\, d(x, y)$ for all $x, y \in X$, for some positive real number c (c is called a Lipschitz constant). (ii) a contractive mapping if $d(T(x), T(y)) < d(x, y)$ for all $x, y \in X$ (iii) a contraction if $d(T(x), T(y)) \leq c\, d(x, y)$ for all $x, y \in X$, for some $0 < c < 1$.

5.3 Theorem (Banach Contraction Principle): If (X, d) is a complete metric space and the mapping $T: X \to X$ is a contraction, then T always has a unique fixed point.

Proof: Fix any x from X. Let us denote this element by x_0. Denote $T(x_0)$ by x_1. For any positive integer n , inductively defining x_n denote $T(x_n)$ by x_{n+1} . The sequence $(x_n)_n$ is a Cauchy sequence, follows from the contractive property of T. X being complete, sequence $(x_n)_n$ is convergent. Suppose $\lim_{n\to\infty} x_n = z$. Then

$$T(z) = T\left(\lim_{n\to\infty} x_n\right) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = z.$$

Uniqueness follows from $d(z, y) = d(T(z), T(y)) \leq c\, d(z, y)$ can not be possible unless $z = y$.

5.4 Example: Suppose $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{1}{2}\arctan x \ \ \forall x \in \mathbb{R}$. Then $|f(x) - f(y)| = \frac{1}{2}|\tan^{-1} x - \tan^{-1} y| \leq \frac{1}{2}|x - y| \ \ \forall x, y \in \mathbb{R}$. So f is a contracting mapping on $\mathbb{R}$. By Banach's theorem $\arctan x = 2x$ for some $x \in \mathbb{R}$.

5.5 Corollary: If (X, d) is a complete metric space and for the mapping $T: X \to X$ there is a positive integer k such that $T^k: X \to X$ is a contraction, then T always has a unique fixed point.

Proof: Since $T^k: X \to X$ is a contraction, from Banach's Contraction Principle T^k always has unique fixed point in $x \in X$. Then $T^k(T(x)) = T\left(T^k(x)\right) = T(x)$ implies $T(x)$ is a fixed point of T^k. From uniqueness property it follows that $T(x) = x$. So x is a fixed point of T. Since any fixed point of T is also a fixed point of T^k x is the unique fixed point of T.

5.6 Example: Suppose f: $\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \frac{\pi}{2} + x - \arctan x \ \ \forall x \in \mathbb{R}$. Then $|f(x) - f(y)| = |x - y - (\tan^{-1} x - \tan^{-1} y)| < |x - y| \ \ \forall x, y \in \mathbb{R}$. So f is a contracting mapping on the complete metric space. But f has no fixed point in $\mathbb{R}$.

5.7 Example: Suppose f: $\mathbb{R}^n \to \mathbb{R}^n$ be a contraction. Let g: $\mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(x) = x - f(x) \ \ \forall x \in \mathbb{R}^n$. Then g is a homeomorphism.

Proof: Since f is a contraction there is some $c > 0$ such that $|f(x) - f(y)| < c|x - y| \ \forall x, y \in \mathbb{R}^n$. Then for $x \neq y$ in $\mathbb{R}^n$ $g(x) - g(y) \neq 0$. So g is injective. To show that g is onto, suppose $u \in \mathbb{R}^n$. Then the mapping $f_u: \mathbb{R}^n \to \mathbb{R}^n$ defined by $f_u(x) = u + f(x)$ is a contraction. By Banach's Contraction Priciple, f_u has a fixed point x in $\mathbb{R}^n$. Then $g(x) = u$ showing that g is onto. From the definition of g it is clear that g and g^{-1} are continuous. Hence g is a homeomorphism.

The contraction mapping principle will now be used to obtain a general result about the existence of a unique solution to a differential equation of the form $\frac{dy}{dx} = f(x, y)$ satisfying certain conditions.

5.8 Definition: Suppose f be a continuous function a rectangle $R = [x_0 - a, x_0 + a]x[y_0 - b, y_0 + b]$. A real valued function φ defined on an interval I is said to be a solution of the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0 \ldots\ldots\ldots\ldots\ldots (1)$ if for $x \in I, f(x, \varphi(x) \in R$ and $\varphi'(x) = f(x, \ \varphi(x))$ with $\varphi(x_0) = y_0$. The following proposition can be easily established.

5.9 Proposition: A function φ is a solution of the initial value problem (1) on an interval I if and only if it is a solution of the integral equation $\varphi(x) = y_0 + \int_{x_0}^{x} f(x, \ \varphi(x))dx$ on $I \ldots\ldots\ldots\ldots\ldots (2)$

5.10 Theorem (Existence and Uniqueness due to E. Picard): Suppose f be a continuous function a rectangle $R = [x_0 - a, x_0 + a]x[y_0 - b, y_0 + b]$ satisfying

$|f(x,y)| \leq K \; \forall \; (x,y) \in R$ and $|f(x,y_1) - f(x,y_2) \leq M$ for some $K > 0, M > 0$.
$\text{Let } h = \min\,(a, \frac{1}{2M},\; b/K)$. Then there is a unique continuously differentiable function $\varphi: (x_0 - h, x_0 + h) \to \mathbb{R}$ on such that $\varphi(x_0) = y_0$ and $\varphi'(x) = f(x, \varphi(x))$ for all $x \in \; (x_0 - h, x_0 + h)$

Proof: Let $I = [x_0 - h, x_0 + h]$ and $X = \{\varphi \in C(I) \colon |\varphi(x) - y_0| \leq b \} \quad \forall \, x \in I$.
Let us define $T \colon X \; \to C(I)$

by $T(\varphi)(x) = y_0 + \int_{x_0}^{x} f\big(x, \; \varphi(x)\big) dx \,, \quad \forall \, x \in I.$

Since $C(I)$ is complete and X is a closed subspace of $C(I)$, X is also complete.
Also for each $\varphi \in X \; |T(\varphi(x) - y_0| = \left|\int_{x_0}^{x} f\big(x, \; \varphi(x)\big) dx\,\right| \leq hK \leq b$ implies $T(X) \subseteq X$. To show that T is a contraction observe that for $\varphi_1, \varphi_2 \; \in$ $X, |T(\varphi_1(x)) - T(\varphi_2(x))| \left|\int_{x_0}^{x} f\big(x, \; \varphi_1(x)\big) dx - \int_{x_0}^{x} f(x, \varphi_2(x)) dx\right| \leq Mh\rho(\varphi_1 \,, \varphi_2)$ where ρ is the supremum metric on $C(I)$. By Banach Contraction Principle T has a unique fixed point $\varphi \in X$. Clearly φ is the desired function.

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