**LECTURE NOTES ON LAPLACE TRANSFORM**

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Laplace transform is an useful mathematical tool to solve the initial value problems. This transformation transforms a particular type of functions **f** of real variable **t** into a related function **F** of a real variable s. When this transform is applied in connection with an initial value problem involving a linear differential equation in an unknown function of **t,** it transform the given differential equation into an algebraic problem involving **s.**

**Def.2.1.1:** *A function f is said to be of exponential order $\alpha$ if there is a positive number* $M$ *such that* $|e^{-\alpha t}f(t)| \leq M \ \ \forall\, t > t_0$ *for some* $t_0 > 0$. *We write* $|f(t)| = o(e^{\alpha t})$.

**Def.2.1.2.** Let f(t) be a real valued function for t > 0. Then the Laplace transform of f(t) is F(s), defined by

$$\mathrm{F(s)} = \int_0^\infty \mathrm{e^{-st}f(t)dt}, \ \ \text{for all values of s for which this integral converges.}$$

We denote the Laplace transform $\mathrm{F(s)}$ by $\mathrm{L\big(f(t)\big)}$. Also then $\mathrm{f(t)}$ will be called inverse Laplace transform of $\mathrm{F(s)}$. We write $\mathrm{f(t) = L^{-1}(F(s))}$.

**Theorem 2.2.1:** Let f be a real valued function such that a) f is piecewise continuous in every closed interval [0, b] (b) f is of exponential order a. Then F(s)=L(f(t)) exists for s > a.

**Properties 2.3.1.**

**i) Linear property:** $\mathrm{L\big(af_1(t) + bf_2(t)\big) = aL\big(f_1(t)\big) + bL\big(f_2(t)\big)}$.

**ii) Translation property:** $\mathrm{L\big(e^{at}f(t)\big) = F(s-a)}$, Where $\mathrm{F(s)}$ is the Laplace transform of $\mathrm{f(t)}$.

**Theorem 2.3.2. i) Suppose F(t) be a continuous function for t $\geq 0$ and be of exponential order $\mathbf{a}$.**

**Suppose $\mathbf{F'(t)}$ be a piecewise continuous function in every closed interval $[\mathbf{0}, \mathbf{t}]$ and of exponential order**

**Then $\mathbf{L\big(F'(t)\big) = pL\big(F(t)\big) - F(0)}$.**

**Proof:** Since $\mathrm{F'(t)}$ is continuous for $\mathrm{t} \geq 0$,

$$\mathrm{L\big(F'(t)\big)} = \int_0^\infty \mathrm{e^{-pt}F'(t)dt} = \lim_{t\to\infty} \mathrm{e^{-pt}F(t) - F(0) + p}\int_0^\infty \mathrm{e^{-pt}F(t)dt} = \mathrm{pL\big(F(t)\big) - F(0)}.$$

**ii) If F(t) is a piecewise continuous function for every closed interval [0, t] and is of exponential order a then $\mathbf{L(tF(t)) = -f'(p), where\ L(F(t)) = f(p). Moreover\ L(t^nF(t)) = (-1)^n\left(\frac{d^n}{dp^n}f(p)\right), for\ n = 1, 2, ..}$**

**Proof:** We know that $f(p) = L(F(t)) = \int_0^\infty e^{-pt}F(t)dt$.

Differentiating with respect to p under the sign of integration, $f'(p) = -\int_0^\infty e^{-pt}tF(t)dt = -L(tF(t))$.

**For the second part, the result can be proved using induction on n.**

**Table 2.4.1:**

| Serial No. | $\mathbf{f(t) = L^{-1}(F(s))}$ | $\mathbf{F(s) = L(f(t))}$ |
|---|---|---|
| 1 | $\mathbf{1}$ | $\mathbf{\frac{1}{s}}$ |
| 2 | $\mathbf{e^{at}}$ | $\mathbf{\frac{1}{s-a}}$ |
| 3 | $\mathbf{\sin bt}$ | $\mathbf{\frac{b}{s^2+b^2}}$ |
| 4 | $\mathbf{\cos bt}$ | $\mathbf{\frac{s}{s^2+b^2}}$ |
| 5 | $\mathbf{\sinh bt}$ | $\mathbf{\frac{b}{s^2-b^2}}$ |
| 6 | $\mathbf{\cosh bt}$ | $\mathbf{\frac{s}{s^2-b^2}}$ |
| 7 | $\mathbf{t^n(n = 1, 2, 3..)}$ | $\mathbf{\frac{n!}{s^{n+1}}}$ |
| 8 | $\mathbf{t^ne^{at}(n = 1, 2, 3..)}$ | $\mathbf{\frac{n!}{(s-a)^{n+1}}}$ |
| 9 | $\mathbf{t\sin bt}$ | $\mathbf{\frac{2bs}{(s^2+b^2)^2}}$ |
| 10 | $\mathbf{t\cos bt}$ | $\mathbf{\frac{s^2-b^2}{(s^2+b^2)^2}}$ |
| 11 | $\mathbf{e^{-at}\sin bt}$ | $\mathbf{\frac{b}{(s+a)^2+b^2}}$ |
| 12 | $\mathbf{e^{-at}\cos bt}$ | $\mathbf{\frac{s+a}{(s+a)^2+b^2}}$ |
| 13 | $\mathbf{\frac{\sin bt - bt\cos bt}{2b^3}}$ | $\mathbf{\frac{1}{(s^2+b^2)^2}}$ |

**Worked out examples:**

**2.5.1. Find the Laplace transform of a function $\mathbf{F(t)}$, defined by $\mathbf{F(t) = \begin{cases} t+1, & 0 \le t \le 2 \\ 3 & t > 2 \end{cases}}$. Also determine $\mathbf{L(F'(t))}$.**

**Ans.** $L(F(t)) = \int_0^\infty e^{-pt}F(t)dt = \int_0^2 (t+1)e^{-pt}\,dt + \int_2^\infty 3e^{-pt}\,dt = \left\{\left[-\frac{(t+1)e^{-pt}}{p}\right]_0^2 - \frac{1}{p^2}[e^{-pt}]_2^\infty\right\} = \frac{1}{p}(1-3e^{-3p}) + \frac{1}{p^2}e^{-2p}.$

**2.5.2. If the Laplace transform of a function $\mathbf{F(t)}$ is denoted by $\mathbf{L(F(t)) = f(s)}$, then show that $\mathbf{L(F(at)) = \frac{1}{a}f\left(\frac{s}{a}\right), a \neq 0}$. Use this to determine $\mathbf{L(e^{3t}\cos t)}$.**

**Ans.** $L(F(at)) = \int_0^\infty e^{-st}F(at)dt = \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t}F(t)dt$ (Substituting at by t) $= \frac{1}{a}f\left(\frac{s}{a}\right),\ a \neq 0.$

Let $A = L(e^{3t}\cos t)$ and $B = L(e^{3t}\sin t)$. Then $A + iB = L(e^{3t}(\cos t + i\sin t)) = L(e^{(3+i)t}) = \frac{1}{3+i}\left(` \frac{1}{\frac{s}{3+i}-1}\right)$. $\left[\text{Since } L(e^t) = \frac{1}{s-1}\right]$.

**2.5.3. Let $\mathbf{F(t)}$ be a periodic function with period $\mathbf{T(>0)}$. Show that $\mathbf{L(F(t)) = \int_0^T \frac{e^{-st}F(t)dt}{1-e^{-sT}}}$.**

**Answer:** By definition $L(F(t)) = \int_0^\infty e^{-st}F(t)dt = \int_0^T e^{-st}F(t)dt + \int_T^\infty e^{-st}F(t)dt = \int_0^T e^{-st}F(t)dt + \int_0^\infty e^{-s(z+T)}F(z+T)dz$. [Substituting $t = z + T$ in the second integral.]. $= \int_0^T e^{-st}F(t)dt + e^{-sT}\int_0^\infty e^{-sz}F(z+T)dz = \int_0^T e^{-st}F(t)dt + e^{-sT}\int_0^\infty e^{-st}F(z)dz = \int_0^T e^{-st}F(t)dt + e^{-sT}L(F(t)).$

So $(1-e^{-sT})L(F(t)) = \int_0^T e^{-st}F(t)dt$. This implies $L(F(t)) = \int_0^T \frac{e^{-st}F(t)dt}{1-e^{-sT}}$ .

**2.5.4.** $\mathbf{F(t) = \begin{cases}(t-1)^2, & t \geq 1\\ 0, & 0 \leq t < 1\end{cases}}$. **Determine $\mathbf{L(F(t))}$.**

**Answer:** $L(F(t)) = \int_0^\infty e^{-pt}F(t)dt = \int_0^1 e^{-pt}F(t)dt + \int_1^\infty e^{-pt}F(t)dt$

$= \int_1^\infty e^{-pt}(t-1)^2dt = \int_0^\infty e^{-p(z+1)}z^2dz = e^{-p}L(z^2) = \frac{e^{-p}2!}{p^3} = \frac{2}{p^3}e^{-p}.$

**2.5.5. State first shifting property of Laplace transform. Find Laplace transform of $t^2e^{at}\sin at$**

**Answer:** Suppose F be such that L(F(t)) exists for s > α. Then for any constant a, $L(e^{at}F(t)) = f(s-a)$ for $s > \alpha + a$, where $f(s)$ denotes $L(F(t))$.

We know that $L(\sin at) = \frac{a}{s^2+a^2}$. Also $L(t^nF(t)) = \frac{(-1)^n d^n}{ds^n}f(s)$, where $f(s)$ is the Laplace transform of $F(t)$.

$$\text{So } L(t^2\sin at) = \frac{(-1)^2d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right) = \frac{d}{ds}\left(-\frac{2as}{(s^2+a^2)^2}\right) = \frac{8as^2(s^2+a^2) - 2a(s^2+a^2)^2}{(s^2+a^2)^4}$$

$$= \frac{6as^2 - 2a^3}{(s^2+a^2)^3}.$$

Therefore by first shifting property $(t^2e^{at}\sin at) = \frac{6a(s-a)^2-2a^3}{((s-a)^2+a^2)^3}$.

**Alternative:** Using first shifting property $L(e^{at}\sin at) = \frac{a}{((s-a)^2+a^2)}$. Then $L(t^2e^{at}\sin at) = \frac{(-1)^2d^2}{ds^2}\left(\frac{a}{(s-a)^2+a^2}\right) = \frac{6a(s-a)^2-2a^3}{((s-a)^2+a^2)^3}$

**2.5.6. Find $\mathbf{L(\sin\sqrt{t})}$ and obtain the value of $\mathbf{L(\frac{\cos\sqrt{t}}{\sqrt{t}})}$.**

**Answer:** For $t > 0, \sin\sqrt{t} = t^{\frac{1}{2}} - \frac{t^{\frac{3}{2}}}{3!} + \frac{t^{\frac{5}{2}}}{5!} - \cdots$

So $L(\sin\sqrt{t}) = L\left(t^{\frac{1}{2}}\right) - \frac{1}{3!}L\left(t^{\frac{3}{2}}\right) + \frac{1}{5!}L\left(t^{\frac{5}{2}}\right) - \cdots$

$$= \frac{\Gamma\left(\frac{3}{2}\right)}{p^{\frac{3}{2}}} - \frac{1}{3!}\left(\frac{\Gamma\left(\frac{5}{2}\right)}{p^{\frac{5}{2}}}\right) + \cdots = \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}\left[1 - \frac{1}{4p} + \frac{1}{2!}\left(\frac{1}{4p}\right)^2 - \cdots\right] = \frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}e^{-\frac{1}{4p}}.$$

Let $F(t) = \sin\sqrt{t}$. Then $F'(t) = \frac{\cos\sqrt{t}}{2\sqrt{t}}$. Therefore $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = 2.L(F'(t)) = 2\{pL(F(t)) - F(0)\}$

$$= 2\left(p.\frac{\sqrt{\pi}}{2p^{\frac{3}{2}}}e^{-\frac{1}{4p}} - 0\right) = \sqrt{\frac{\pi}{p}}e^{-\frac{1}{4p}}.$$

**2.5.7. State a set of sufficient conditions for existence of Laplace transform. Show that Laplace transform is a linear operator. If f is the Laplace transform of F, determine the Laplace transform of G defined by $\mathbf{G(t) = \begin{cases} 0, & 0 < t < a \\ F(t-a) & t > a \end{cases}}$.**

**Answer:** If F(t ) is a piecewise continuous function in every closed interval [0, t] for every t>= 0 and is of exponential order a, then Laplace transform of F(t) exists for p > a.

Suppose F(t) and G(t) have Laplace transform f(p) and g(p) respectively, for p > a and let c and d be two scalar. Then $L(cF(t) + dG(t)) = \int_0^\infty e^{-pt}(cF(t) + dG(t)) = c\int_0^\infty e^{-pt}F(t) + d\int_0^\infty e^{-pt}G(t) = cf(p) + dg(p)$ Thus L is a linear operator.

Given that $f(p) = \int_0^\infty e^{-pt}F(t)dt$.

$$\text{Now } g(p) = \int_0^\infty e^{-pt}G(t)dt. = \int_a^\infty e^{-pt}F(t-a)dt = \int_0^\infty e^{-p(z+a)}F(z)dz = e^{-ap}\int_0^\infty e^{-pt}F(t)d\,t = e^{-ap}f(p).$$

**2.5.8. Use first shifting property of Laplace transform (or otherwise ) to evaluate $\mathbf{L^{-1}\left(\frac{s-10}{s^2-4s+20}\right)}$.**

**Answer:** Here $L^{-1}\left(\frac{s-10}{s^2-4s+20}\right) = L^{-1}\left(\frac{s-2-8}{(s-2)^2+16}\right) = L^{-1}\left(\frac{s-2}{(s-2)^2+16}\right) - 2L^{-1}\left(\frac{4}{(s-2)^2+4^2}\right) = e^{2t}\cos 4t - 2\,e^{2t}\sin 4t$

**2.5.9. Using shifting property of Laplace transform (or otherwise ) to evaluate $\mathbf{L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right)}$.**

**Answer:** Here $L^{-1}\left(\frac{6s-4}{s^2-4s+20}\right) = L^{-1}\left(\frac{6(s-2)+8}{(s-2)^2+16}\right) = 6L^{-1}\left(\frac{s-2}{(s-2)^2+16}\right) + 2L^{-1}\left(\frac{4}{(s-2)^2+4^2}\right) = 6e^{2t}\cos 4t + 2\,e^{2t}\sin 4t$

**2.5.10. State the Convolution theorem on Laplace transform and using this theorem find $\mathbf{L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right)}$**

**Answer: If F(t) and $\mathbf{G(t)}$ are two piecewise continuous functions in every closed interval $\mathbf{[0,t]}$ and f(p) and g(p) be their respective Laplace transforms then f(p).g(p) is the Laplace transform of F$*$ G**

$$\mathbf{where\ (F * G)(t) = \int_0^t F(t-u)G(u)du\,.}$$

Let $F(t) = L^{-1}\left(\frac{1}{s+1}\right)$ and $G(t) = L^{-1}\left(\frac{1}{(s^2+1)}\right)$.

Then $F(t) = e^{-t}$ and $G(t) = \sin t$.

Now $(F * G)(t) = \int_0^t F(t-u)G(u)du = \int_0^t e^{-(t-u)}\sin u\, du = e^{-t}\int_0^t e^u \sin u\, du$.

Let $I = \int_0^t e^u \sin u\, du$. Then $I = \sin u\, e^u|_0^t - \int_0^t e^u \cos u\, du = e^t \sin t - [\cos u\, e^u]_0^t - I$.

Then $2I = e^t(\sin t - \cos t) + 1$. So $(F * G)(t) = \frac{1}{2}(\sin t - \cos t) + \frac{e^{-t}}{2}$

By convolution theorem $L^{-1}\left(\frac{1}{(s+1)(s^2+1)}\right) = \frac{1}{2}(\sin t - \cos t) + \frac{e^{-t}}{2}$

**2.5.11. State the Convolution theorem on Laplace transform and using this theorem prove that $\boldsymbol{L^{-1}\left(\frac{1}{(p+2)^2(p-2)}\right) = \frac{1}{16}(e^{2t} - 4te^{-2t} - e^{-2t})}$**

**Answer:** Convolution theorem is already stated in previous example.

For the second part let $F(t) = L^{-1}\left(\frac{1}{(p+2)^2}\right)$ and $G(t) = L^{-1}\left(\frac{1}{p-2)}\right)$.

Then $F(t) = te^{-2t}$ and $G(t) = e^{2t}$

Now $(F*G)(t) = \int_0^t F(u)G(t-u)du = \int_0^t ue^{-2u}\, e^{2(t-u)}\, du = e^{2t}\int_0^t ue^{-4u}du = e^{2t}\{\left[\frac{u}{-4}e^{-4u}\right]_0^t +$
$\frac{1}{4}\int_0^t e^{-4u}du\} = e^{2t}\left\{\frac{t}{-4}e^{-4t} - \frac{1}{16}(e^{-4t}-1)\right\} = \frac{e^{2t}}{16} - \frac{e^{-2t}}{16} - \frac{te^{-2t}}{4}$

So by convolution theorem $L^{-1}\left(\frac{1}{(p+2)^2(p-2)}\right) = \frac{1}{16}(e^{2t} - 4te^{-2t} - e^{-2t})$.

**2.5.12.Evaluate $\mathbf{L^{-1}\left(\frac{1}{(p-3)^2(p+4)}\right)}$**

**Answer:** Let $f(p) = \frac{1}{(p-3)^2(p+4)}$. Then $f(p) = \frac{A}{p-3} + \frac{B}{(p-3)^2} + \frac{C}{p+4}$. Then $1 = A(p-3)(p+4) + B(p+4) + C(p-3)^2$. Comparing the like powers in p, we get $7B = 1, 49C = 1, -12A + 4B + 9C = 1$. So $B = \frac{1}{7}, C = \frac{1}{49}, A = -\frac{1}{49}$.

Then $L^{-1}\left(\frac{1}{(p-3)^2(p+4)}\right) = -\frac{1}{49}L^{-1}\left(\frac{1}{p-3}\right) + \frac{1}{7}L^{-1}\left(\frac{1}{(p-3)^2}\right) + \frac{1}{49}L^{-1}\left(\frac{1}{p+4}\right)$

$= -\frac{1}{49}(e^{3t} - e^{-4t}) - \frac{1}{7}te^{3t}$.

**2.5.13. Evaluate $\mathbf{L^{-1}\left(\frac{p^2}{(p^2+4)^2}\right)}$**

**Answer:** Let $f(p) = \frac{1}{(p^2+4)}$. Then $F(t) = L^{-1}(f(p)) = \frac{1}{2}L^{-1}\left(\frac{2}{p^2+2^2}\right) = \frac{1}{2}\sin 2t$.

Now $L(tF(t)) = -\left(\frac{d}{dp}f(p)\right) = \frac{2p}{(p^2+2^2)^2} = g(p)$, say. So $G(t) = L^{-1}\left(\frac{2p}{(p^2+4)^2}\right) = \frac{t}{2}\sin 2t$ .

Then $L(G'(t)) = pL(G(t)) - G(0) = \frac{2p^2}{(p^2+4)^2}$. Therefore $L^{-1}\left(\frac{p^2}{(p^2+4)^2}\right) = \frac{1}{2}G'(t)$

$$= \left(\frac{1}{4}\right)\frac{d}{dt}(t\sin 2t) = \frac{1}{4}(\sin 2t + 2t\cos 2t)$$

**2.5.14. Find $\mathbf{L^{-1}\left(\frac{p}{p^4+4}\right)}$**

**Answer:** Let f(p)= $\frac{p}{p^4+4}$. Then $f(p) = \frac{p}{(p^2-2p+2)(p^2+2p+2)} = \frac{\frac{1}{4}\left((p^2+2p+2)-(p^2-2p+2)\right)}{(p^2-2p+2)(p^2+2p+2)} = \frac{1}{4}\left[\frac{1}{(p^2-2p+2)} - \frac{1}{(p^2+2p+2)}\right] = \frac{1}{4}\left[\frac{1}{(p-1)^2+1} - \frac{1}{(p+1)^2+1}\right]$.

So $L^{-1}(f(p)) = \frac{1}{4}[e^t\sin t - e^{-t}\sin t]$.

**2.5.14.Find $\mathbf{L^{-1}\left(\frac{2p^2-6p+5}{p^3-6p^2+11p-6}\right)}$**

**Answer:** Let f(p)= $\frac{2p^2-6p+5}{p^3-6p^2+11p-6}$. Then $f(p)=\frac{2p^2-6p+5}{(p-1)(p^2-5p+6)}=\frac{2p^2-6p+5}{(p-1)(p-2)(p-3)}=\frac{A}{p-1}+\frac{B}{p-2}+\frac{C}{p-3}$.

Equating like powers of p we get $2A=1,-B=1,2C=5$. So $L^{-1}\big(f(p)\big)=\frac{1}{2}e^t-e^{2t}+\frac{5}{2}e^{3t}$.

**2.5.15. State the Convolution theorem for inverse Laplace transform and using it find $\mathbf{L^{-1}\left(\frac{p+1}{(p^2+2p+2)^2}\right)}$.**

**Answer:** If $L^{-1}\big(f(p)\big)=F(t)$ and $L^{-1}\big(g(p)\big)=G(t)$ be the inverse Laplace transforms of f(p)and g(p) respectively, then the inverse Laplace transform of f(p). g(p)is $(F*G)(t)$ where

$(F*G)(t)=\int_0^t F(u)G(t-u)du$.

Let f(p)=$\frac{p+1}{p^2+2p+2}$ and $g(p)=\frac{1}{p^2+2p+2}$

Then the inverse Laplace transform of f(p)abd g(p)are $F(t)=L^{-1}\left(\frac{p+1}{p^2+2p+2}\right)=L^{-1}\left(\frac{p+1}{(p+1)^2+1}\right)=$ $e^{-t}\cos t$ and $G(t)=L^{-1}\left(\frac{1}{p^2+2p+2}\right)=L^{-1}\left(\frac{1}{(p+1)^2+1}\right)=e^{-t}\sin t$

Now $(F*G)(t)=\int_0^t F(u)G(t-u)du=\int_0^t e^{-(t-u)}\sin(t-u)\ e^{-u}\cos u\ du=e^{-t}\int_0^t \sin(t-u)\cos u\ du$ .

$$=\frac{e^{-t}}{2}\int_0^t\{\sin t+\sin(t-2u)\}du=\frac{e^{-t}}{2}\left\{t\sin t+\left[\frac{-\cos(t-2u)}{2}\right]_0^t\right\}=\frac{e^{-t}}{2}t\sin t$$

Thus $L^{-1}\left(\frac{p+1}{(p^2+2p+2)^2}\right)=\frac{e^{-t}}{2}t\sin t$

**2.5.16. Using convolution theorem, find $\mathbf{L^{-1}\left(\frac{1}{p(p^2+4)^2}\right)}$.**

**Answer:** Suppose $f(p)=\frac{p}{p^2+4}$ and $g(p)=\frac{1}{p^2(p^2+4)}$

Then the inverse Laplace transform of f(p) and g(p) are respectively $F(t)=\cos 2t$ $G(t)=$ $L^{-1}\left(\frac{1}{p^2(p^2+4)}\right)=L^{-1}\left(\frac{1}{4}\left(\frac{1}{p^2}-\frac{1}{p^2+4}\right)\right)=\frac{1}{4}\left(t-\frac{1}{2}\sin 2t\right)$

Then $(F*G)(t)=\int_0^t F(u)G(t-u)du$

$$=\frac{1}{4}\int_0^t\left(t-\frac{1}{2}\sin 2t\right)\cos(2t-2u)\,du$$

$$=\ \frac{1}{4}\left\{\left[-\frac{1}{2}\sin(2t-2u)\right]_0^t+\frac{1}{2}\int_0^t\sin(2t-2u)du-\frac{1}{4}\int_0^t(\sin(4t-2u)+\sin 2u)du\right\}$$

$$=\frac{1}{8}\sin 2t+\frac{1}{16}[\cos(2t-2u)]_0^t-\frac{1}{8}[\cos(4t-2u)]_0^t+\frac{1}{8}[\cos 2u]_0^t$$

$$=\frac{1}{8}\sin 2t+\frac{1}{16}-\frac{1}{16}\cos 2t\ -\frac{1}{8}\cos 2t+\frac{1}{8}\cos 4t+\frac{1}{8}\cos 2t-\frac{1}{8}$$

$$=\frac{1}{8}(\sin 2t-\frac{1}{2}\cos 2t+\cos 4t-\frac{1}{2})$$

So by convolution theorem $L^{-1}\left(\frac{1}{p(p^2+4)^2}\right)=\frac{1}{8}(\sin 2t-\frac{1}{2}\cos 2t+\cos 4t-\frac{1}{2})$

**2.5.17. If the Laplace transform of a function F(t) is defined by L(F(t)) and if L(F(t))=f(p), then show that L(F(at))=** $\boldsymbol{\frac{1}{a}f\left(\frac{p}{a}\right), a\neq 0}$**. Using this property, find** $\boldsymbol{L(F(2t))}$ **where** $\boldsymbol{L(F(t))=\frac{p^2-p+1}{(2p+1)^2(p-1)}}$

**Answer:** Given that $L(F(t))=\int_0^\infty e^{-pt}F(t)dt$.

Now $L(F(at))=\int_0^\infty e^{-pt}F(at)dt$

$$=\frac{1}{a}\int_0^\infty e^{-\frac{pz}{a}}F(z)dz$$

$=\frac{1}{a}f\left(\frac{p}{a}\right), a\neq 0.$

Here $L(F(t))=\frac{p^2-p+1}{(2p+1)^2(p-1)}$.

So $L(F(2t))=\frac{\frac{1}{2}\left(\left(\frac{p}{2}\right)^2-\frac{p}{2}+1\right)}{\left(2.\frac{p}{2}+1\right)^2(\frac{p}{2}-1)}=\frac{\frac{1}{4}(p^2-2p+4)}{(p+1)^2(p-2)}$.

**2.5.18. Find inverse Laplace transform of** $\boldsymbol{\frac{s+4}{s(s-1)(s^2+4)}}$.

**Answer:** Let $\frac{s+4}{s(s-1)(s^2+4)}=\frac{A}{s}+\frac{B}{s-1}+\frac{Cs+D}{s^2+4}$

Then $s+4=A(s-1)(s^2+4)+Bs(s^2+4)+Cs^2(s-1)+Ds(s-1)$

Comparing like powers in s we get $-4A=4, 5B=5, A+B+C=0, -A-C+D=0$

So $A=-1, B=1, C=0, D=-1$.

Then $L^{-1}\left(\frac{s+4}{s(s-1)(s^2+4)}\right) = -L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s-1}\right) - L^{-1}\left(\frac{1}{s^2+4}\right)$

$= e^t - 1 - \frac{1}{2}\sin 2t$

**2.5.19. Find inverse Laplace transform of** $\frac{\mathbf{5p+3}}{\mathbf{(p-1)(p^2+2p+5)}}$.

**Answer:** Let $f(p) = \frac{5p+3}{(p-1)(p^2+2p+5)} = \frac{A}{p-1} + \frac{Bp+C}{p^2+2p+5}$, say.

Then $5p+3 = A(p^2+2p+5) + (Bp+C)(p-1)$.

Equating like powers, $8A = 8,\ A+B = 0, 5A - C = 3$.

This implies $A = 1, B = -1, C = 2$.

So $f(p) = \dfrac{1}{p-1} + \dfrac{-p+2}{((p+1)^2+2^2)} = \dfrac{1}{p-1} - \dfrac{p+1}{((p+1)^2+2^2)} + \dfrac{3}{2}\dfrac{2}{((p+1)^2+2^2}$

$L^{-1}\left(\dfrac{5p+3}{(p-1)(p^2+2p+5)}\right) = e^t - e^{-t}(\cos 2t\ - \dfrac{3}{2}\sin 2t)$

**2.6.Solving Differential equation by Laplace method.**

**2.6.1. Example : Using Laplace transforms solve the following differential equation**

$$(\mathbf{D^2+4D+8})\mathbf{y} = \mathbf{0}, \qquad \mathbf{y(0) = 2,\ y'(0) = 2}$$

**Answer:** Here the problem may be written as $y'' + 4y' + 8y = 0$ … … … … … … … … … … … (1)

$y(0) = 2,\ y'(0) = 2$……………………………………….(2)

Applying Laplace transforms to both sides of (1) we get

$L(y'') + 4L(y') + 8L(y) = 0$

or, $p^2L(y) - py(0) - y'(0) + 4pL(y) - 4y(0) + 8L(y) = 0$

or, $(p^2+4p+8)L(y) = 2p + 10$

So $L(y) = \frac{2p+10}{(p^2+4p+8)}$

Then $y = L^{-1}\left(\frac{2p+10}{(p^2+4p+8)}\right) = L^{-1}\left(\frac{2(p+2)}{(p^2+4p+8)} + \frac{6}{(p^2+4p+8)}\right) = 2e^{-2t}\cos 2t + 3e^{-2t}\sin 2t$ .

Which is the required solution.

**2.6.2. Using Laplace transform, solve**

$$\mathbf{\frac{2d^2y}{dx^2} + \frac{5dy}{dx} + 2y = e^{-2x}, when\ y(0) = 1\ \ and \frac{dy}{dx}(0) = 1}.$$

**Answer:** Given equation is $\frac{2d^2y}{dx^2} + \frac{5dy}{dx} + 2y = e^{-2x}$ ……………………..(1)

Initial conditions are $y(0) = 1$ and $\frac{dy}{dx}(0) = 1$…………………..(2)

Applying Laplace transforms both sides of (1) we get

$$2L(\frac{d^2y}{dx^2}) + 5L(\frac{dy}{dx}) + 2L(y) = L(e^{-2x})$$

or , $2\left(p^2L(y) - py(0) - y'(0)\right) + 5pL(y) - 5y(0) + 2L(y) = \frac{1}{p+2}$

or , $(2p^2 + 5p + 2)L(y) - 2p - 7 = \frac{1}{p+2}$

or , $L(y) = \frac{2p^2+11p+15}{(2p^2+5p+2)(p+2)}$

**2.6.3. Using Laplace transform, solve**

$$\mathbf{\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 10y = 0, when\ y(0) = 1, y\left(\frac{\pi}{4}\right) = \sqrt{2}.}$$

**Answer:** Given equation is $\frac{d^2y}{dx^2} + \frac{6dy}{dx} + 10y = 0$ ……………………..(1)

Initial conditions are $y(0) = 1$ and $y\left(\frac{\pi}{4}\right) = \sqrt{2}$…………………..(2)

Applying Laplace transforms both sides of (1) we get

$$L(\frac{d^2y}{dx^2}) + 6L(\frac{dy}{dx}) + 10L(y) = 0$$

or , $\left(p^2L(y) - py(0) - y'(0)\right) + 6pL(y) - 6y(0) + 10L(y) = 0$

or , $(p^2 + 6p + 10)L(y) - p - 6 - c = 0,$ where $y'(0) = c$, say

or , $L(y) = \frac{p+6+c}{p^2+6p+10} = \frac{p+3}{p^2+6p+10} + \frac{3+c}{p^2+6p+10}$

So $y = e^{-3x}(\cos x + (3 + c)\sin x)$

From (2), $\sqrt{2} = e^{-3\frac{\pi}{4}}\frac{(4 + c)}{\sqrt{2}}$

$\therefore\ c = 2e^{3\frac{\pi}{4}} - 4$

So the required solution is $y = e^{-3x}\left(\cos x + \left(2e^{3\frac{\pi}{4}} - 1\right)\sin x\right)$

**2.6.4. Solve by using Laplace transform,**

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} + 2y = 10\cos x\text{, given that } y(0) = 0, y'(0) = 7$$

**2.6.5. Using Laplace transform, solve**

$$\frac{d^2y}{dt^2} + \frac{2dy}{dt} + y = 3te^{-t}, t > 0$$

where $y(0) = 4$ and $y'(0) = 2$.

**2.6.6. Using Laplace transform, solve**

$$\frac{d^2x}{dt^2} + \frac{2dx}{dt} + x = 3te^{-t}, t > 0$$

with $x(0) = 4$ and $x'(0) = 2$.

**2.6.7. Using Laplace transform, solve**

$$\frac{d^2y}{dx^2} - \frac{5dy}{dx} + 6y = 5\text{, when } y(0) = 0 = y(1)$$

**2.6.8. Using Laplace transform, solve**

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} = e^{3t}$$