1. 1 A sequence is…

(a) an ordered list of objects.

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \cdots$$

(b) A function whose domain is a set of integers

Domain: 1, 2, 3, 4, …, *n*…

$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \qquad \downarrow$$

Range $a_1, a_2, a_3, a_4, \ldots\ a_n \ldots$

{(1, 1), (2, ½), (3, ¼), (4, 1/8) ….}

Finding patterns

Describe a pattern for each sequence. Write a formula for the nth term

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \ldots \qquad \frac{1}{2^{n-1}}$$

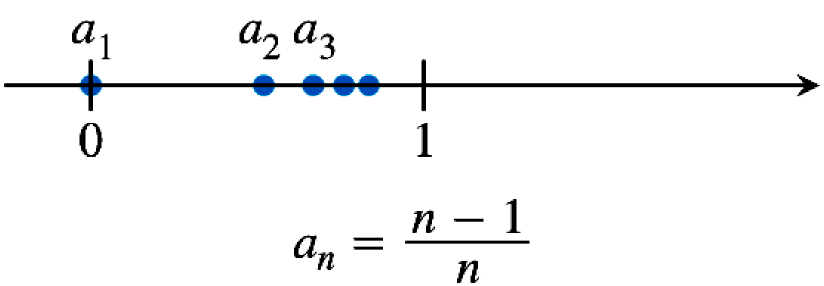
$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120} \ldots \qquad \frac{1}{n!}$$

$$\frac{1}{4}, \frac{4}{9}, \frac{9}{16}, \frac{16}{25}, \frac{25}{36} \ldots \qquad \frac{n^2}{(n+1)^2}$$

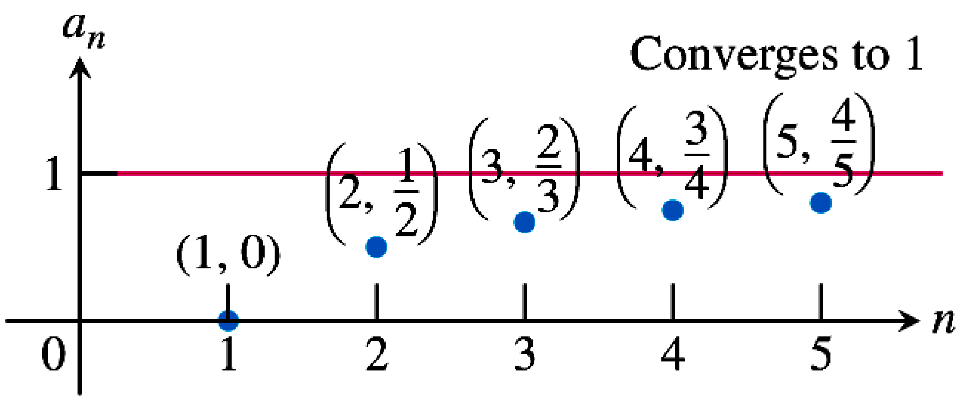
Write the first 5 terms for $a_n = \dfrac{n-1}{n}$

$$0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \cdots \frac{n-1}{n} \cdots$$

On a number line



As a function

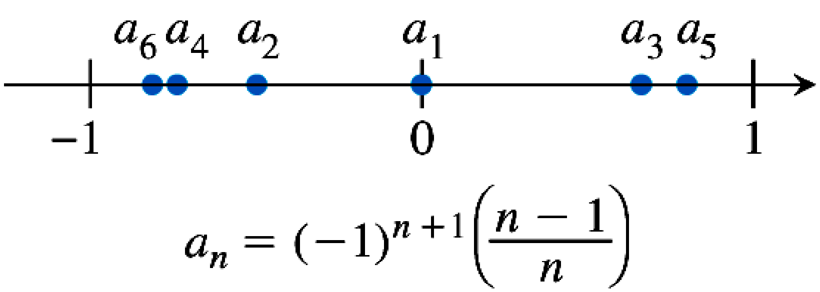


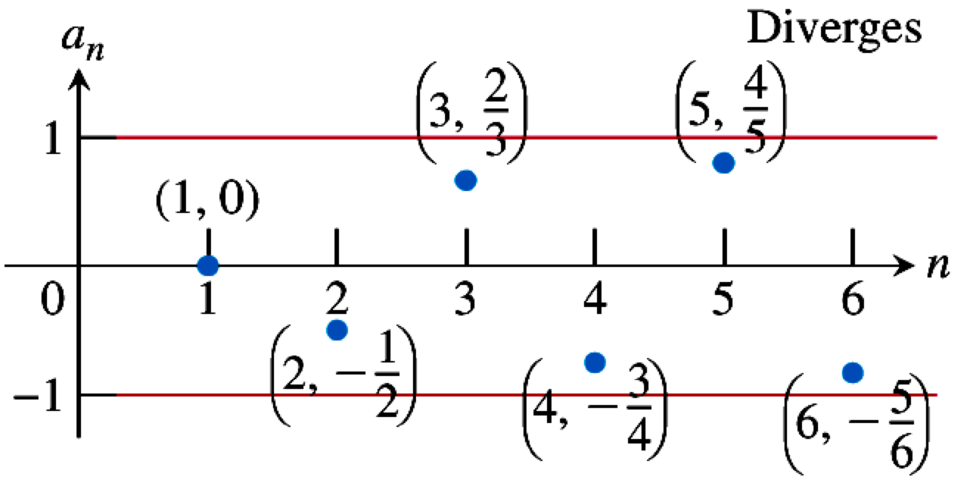
The terms in this sequence get closer and closer to 1. The sequence CONVERGES to 1.

Write the first 5 terms

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$

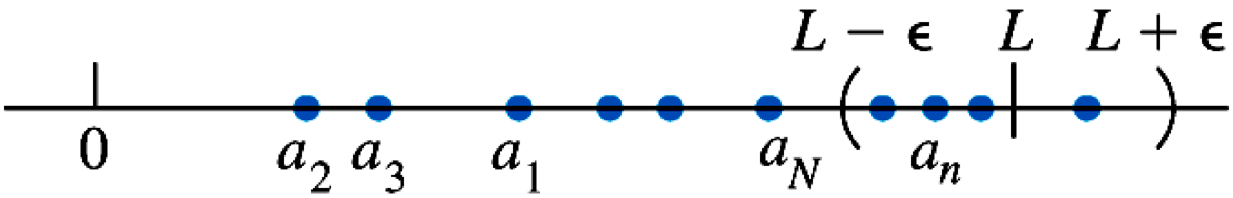
$$0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}. - + - \ldots \frac{n-1}{n} \ldots$$

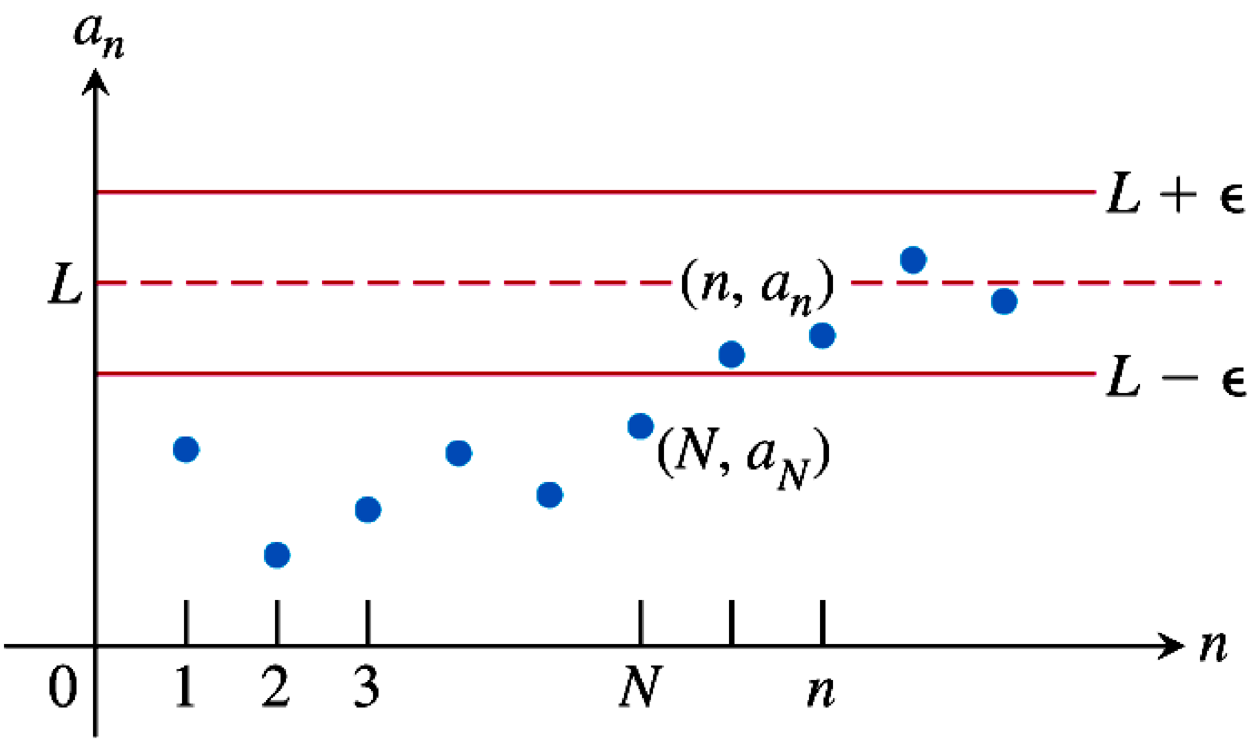




The terms in this sequence do not get close to
Any single value. The sequence Diverges

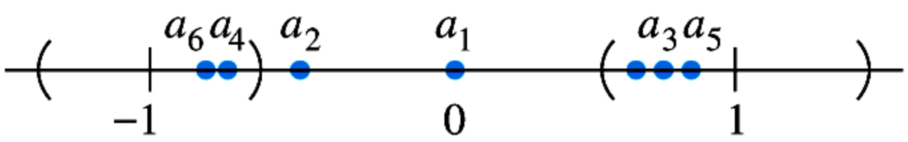
y= L is a horizontal asymptote when sequence converges to L.





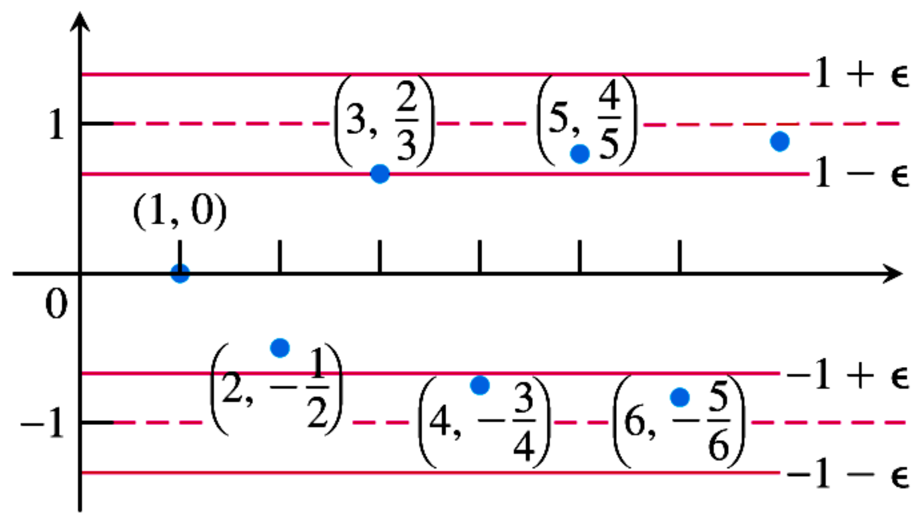
A sequence that diverges

$$a_n = \frac{(-1)^{n+1}(n-1)}{n}$$



$$a_n = (-1)^{n+1}\left(\frac{n-1}{n}\right)$$

Neither the ϵ-interval about 1 nor the ϵ-interval about -1 contains all a_n satisfying $n \geq N$ for some N.



12.2 Infinite Series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$$

Represents the sum of the terms in a sequence.

We want to know if the series converges to a single value i.e. there is a finite sum.

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + 1 + \ldots$$

The series diverges because $s_n = n$. Note that the **Sequence** $\{1\}$ converges.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \ldots$$

Partial sums of $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \frac{1}{30} + \dots$

$$s_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$s_2 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} = \frac{2}{3}$$

$$s_3 = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} = \frac{3}{4}$$

and

$$s_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \frac{1}{n(n+1)} = \frac{n}{n+1}$$

If the sequence of partial sums converges, the series converges

$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6} \dots \frac{n}{n+1} \dots$ Converges to 1 so **series** converges.

Finding sums $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Can use partial fractions to rewrite

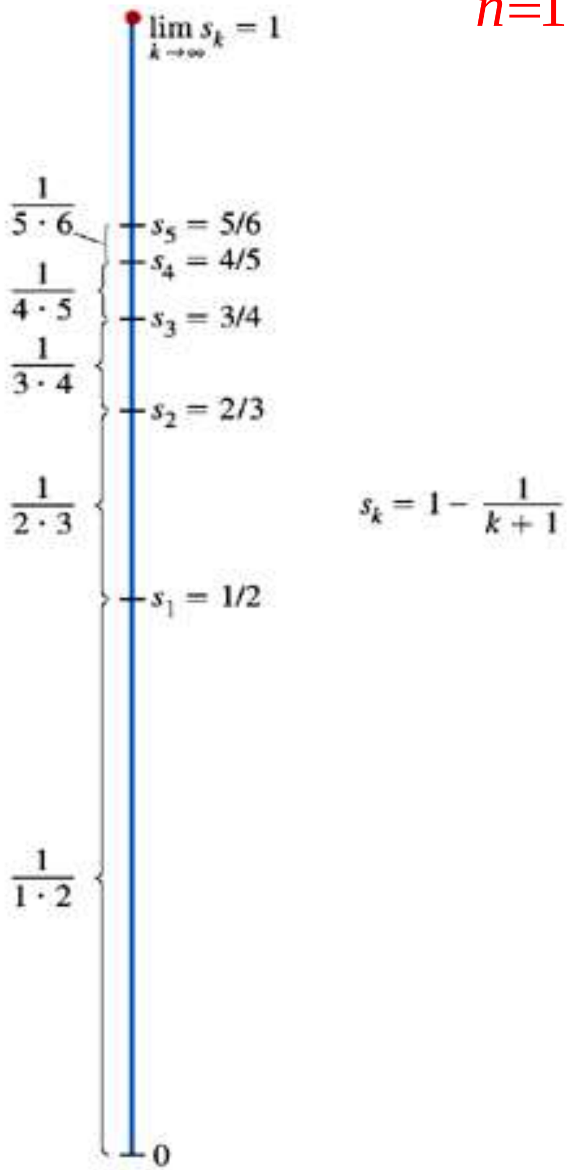
$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} =$$

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots \frac{1}{n} - \frac{1}{n+1} + \ldots$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} (1 - \frac{1}{n+1}) = 1$$

The partial sums of the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

Limit



The sum of a geometric series

$$s_n = a + ar + ar^2 + ar^3 + \dots ar^{n-1}$$

Sum of n terms

$$rs_n = ar + ar^2 + ar^3 + \dots ar^n$$

Multiply each term by r

$$s_n - rs_n = a - ar^n$$

subtract

$$s_n = \frac{a - ar^n}{1-r} = \frac{a(1-r^n)}{1-r}, r \neq 1$$

$$if \ |r| < 1, \qquad r^n \to 0 \ as \ n \to \infty$$

Geometric series converges to $\boxed{s_n = \frac{a}{1-r}, |r| < 1}$

If r>1 the geometric series diverges.

Series known to converge or diverge

1. A geometric series with $|r| < 1$ converges

2. A repeating decimal converges

3. Telescoping series converge

A necessary condition for convergence:
Limit as *n* goes to infinity for nth term in sequence is 0.

nth term test for divergence:
If the limit as n goes to infinity for the nth term is not 0, the series DIVERGES!

Convergence or Divergence?

$$\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+2}$$

$$\sum_{n=1}^{\infty} (1.075)^{n}$$

$$\sum_{n=1}^{\infty} \frac{4}{2^{n}}$$

A sequence in which each term is less than or equal to the one before it is called a **monotonic non-increasing** sequence. If each term is greater than or equal to the one before it, it is called **monotonic non-decreasing.**

A monotonic <u>sequence</u> that is bounded
Is convergent.

A <u>series</u> of non-negative terms converges
If its partial sums are bounded from above.

12. 3 The Integral Test

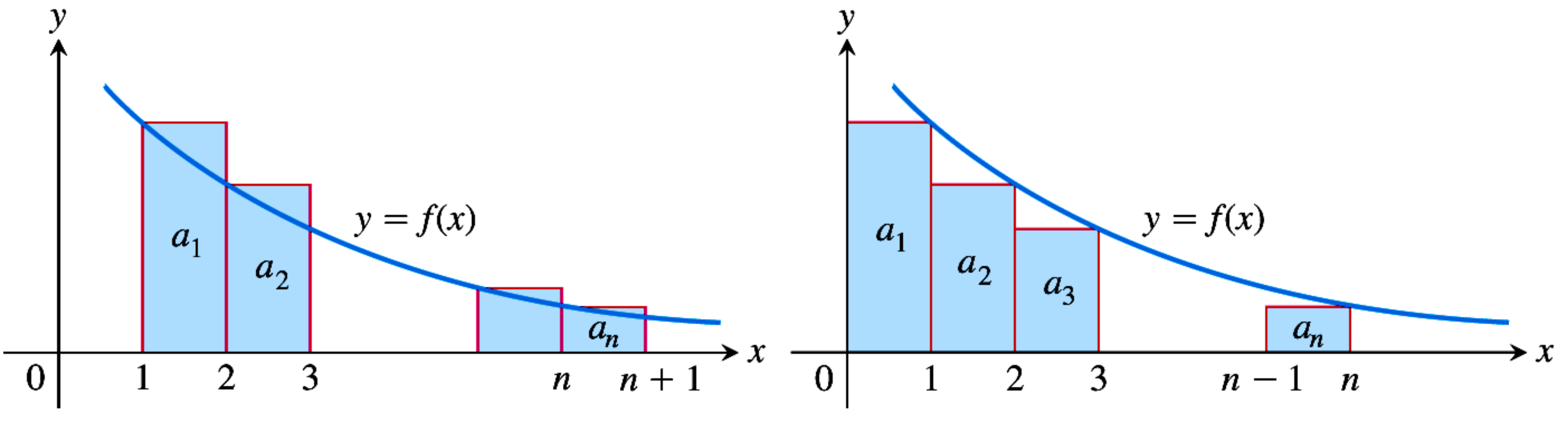
Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$ where f is a continuous positive, decreasing function of x for all $x \geq N$. Then the series and the corresponding integral shown **both converge** of **both diverge**.

$$\sum_{n=N}^{\infty} a_n \qquad \int_N^{\infty} f(x)\,dx$$

$f(n)$ $\qquad$ $f(x)$

The series and the integral both converge or both diverge

Area in rectangle corresponds to term in sequence



(a) (b)

Exact area under curve is between

If area under curve is finite, so is area in rectangles

If area under curve is infinite, so is area in rectangles

Using the Integral test

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1} \qquad a_n = f(n) = \frac{n}{n^2+1} \qquad f(x) = \frac{x}{x^2+1}$$

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{1}{2}\int_1^{b} \frac{2x}{x^2+1}\,dx = \lim_{b\to\infty} \left[\ln(x^2+1)\right]_1^b$$

$$\lim_{b\to\infty}\left(\ln(b^2+1) - \ln 2\right) = \infty$$

The improper integral diverges

Thus the series diverges

Harmonic series and p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$ Is called a p-series

A p-series converges if p > 1 and diverges If p < 1 or p = 1.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots\frac{1}{n} + \ldots$$

Is called the harmonic series and it diverges since p =1.

Limit Comparison test

$$\lim_{x\to\infty}\frac{a_n}{b_n}=c, \qquad 0<c<\infty$$

Then the following series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

both converge or both diverge:

$\lim_{x\to\infty}\frac{a_n}{b_n}=0$ and $\sum_{n=1}^{\infty} b_n$ Converges then $\sum_{n=1}^{\infty} a_n$ Converges

$\lim_{x\to\infty}\frac{a_n}{b_n}=\infty$ and $\sum_{n=1}^{\infty} b_n$ Diverges then $\sum_{n=1}^{\infty} a_n$ Diverges

Alternating Series

A series in which terms alternate in sign

$$\sum_{n=1}^{\infty}(-1)^n a_n$$ or $$\sum_{n=1}^{\infty}(-1)^{n+1} a_n$$

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{2^n}=\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\ldots$$

$$\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}=-1+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\ldots$$

Alternating Series Test

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - a_2 + a_3 - a_4 + \ldots$$

Converges if:

✓ a_n is always positive

✓ $a_n \geq a_{n+1}$ for all $n \geq N$ for some integer N.

✓ $a_n \to 0$

If any one of the conditions is not met, the Series diverges.

Absolute and Conditional Convergence

- A series $\sum_{n=N}^{\infty} a_n$ is **absolutely convergent** if the corresponding series of absolute values $\sum_{n=N}^{\infty} |a_n|$ converges.

- A series that converges but does not converge absolutely, converges conditionally.

- Every absolutely convergent series converges. (Converse is false!!!)

The Ratio Test

Let $\sum_{n=N}^{\infty} a_n$ be a series with positive terms and

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho$$

Then

- The series converges if $\rho < 1$
- The series diverges if $\rho > 1$
- The test is inconclusive if $\rho = 1$.

The Root Test

Let $\sum_{n=N}^{\infty} a_n$ be a series with non-zero terms and

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

Then

- The series converges if L< 1
- The series diverges if L > 1 or is infinite
- The test is inconclusive if L= 1.

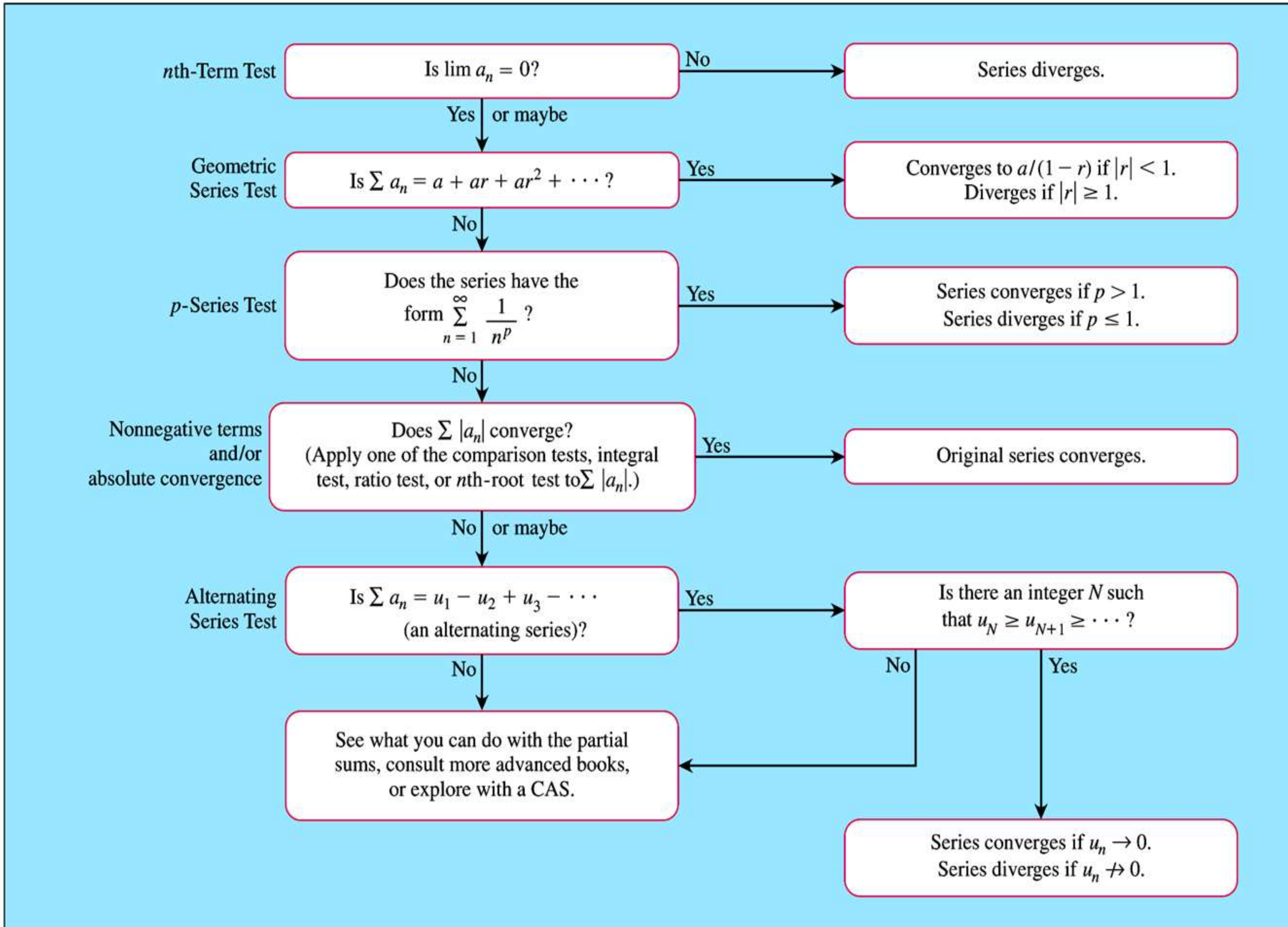
Convergence or divergence?

$$\sum_{n=1}^{\infty} \frac{2^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2 2^{n+1}}$$

$$\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$$

Procedure for determining Convergence



Power Series (infinite polynomial in *x*)

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots\ldots c_n x^n \ldots$$

Is a power series centered at $x = 0$.

and

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots\ldots c_n (x-a)^n \ldots$$

Is a power series centered at $x = a$.

Examples of Power Series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \ldots$$

Is a power series centered at $x = 0$.

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n}(x+1)^n = 1 - \frac{1}{3}(x+1) + \frac{1}{9}(x+1)^2 - \ldots\ldots \frac{1}{3^n}(x+1)^n \ldots$$

Is a power series centered at $x = -1$.

Geometric Power Series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots x^n$$

$$a = 1 \quad and \quad r = x$$

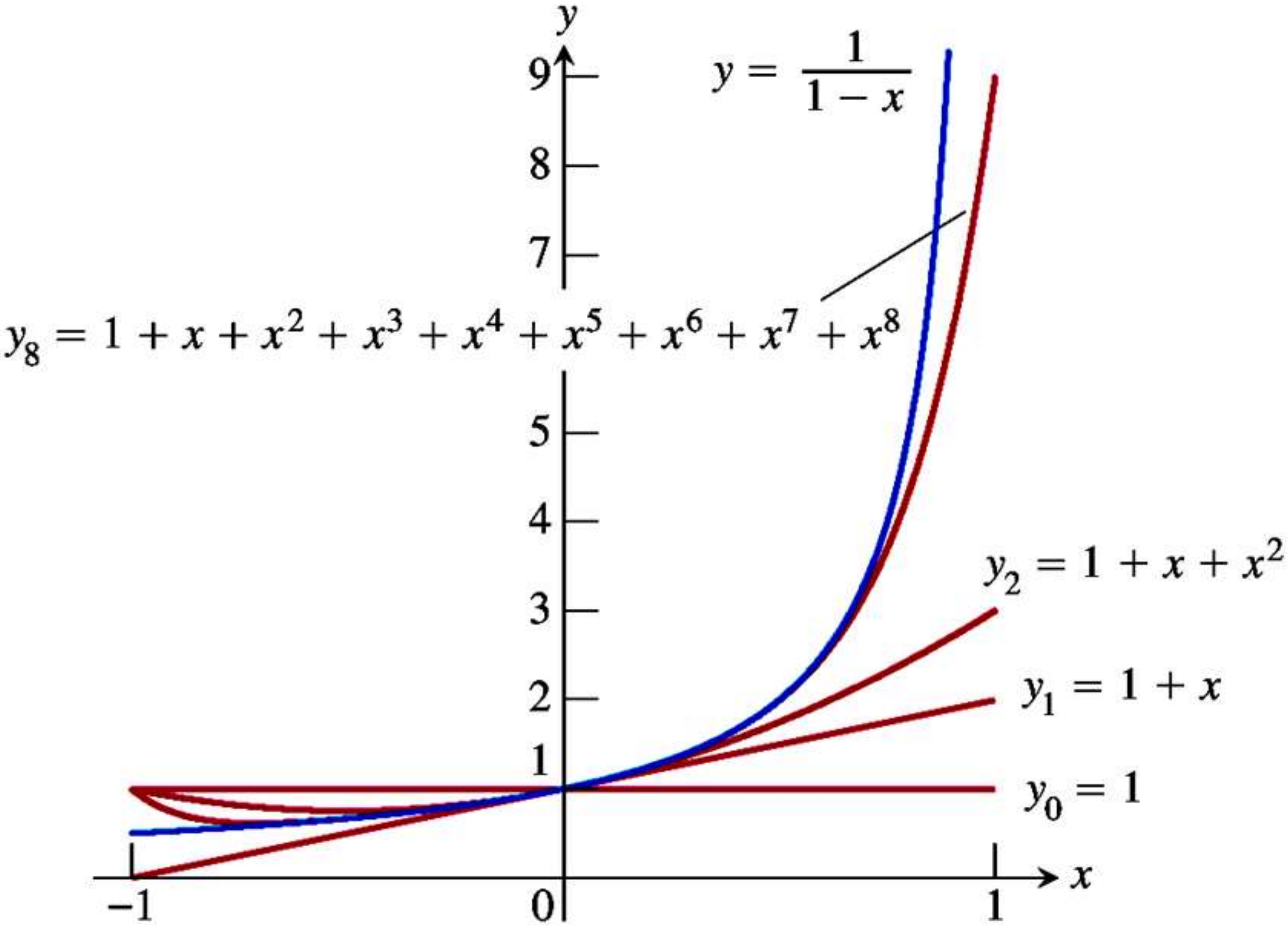
$$S = \frac{a}{1-r} = \frac{1}{1-x}, \quad |x| < 1$$

$$P_1 = 1 + x$$

$$P_2 = 1 + x + x^2$$

$$P_3 = 1 + x + x^2 + x^3$$

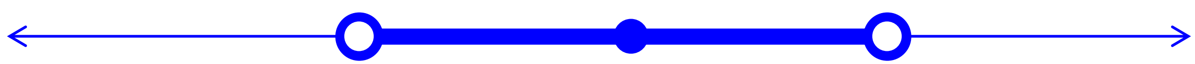
The graph of f(x) = 1/(1-x) and four of its polynomial approximations



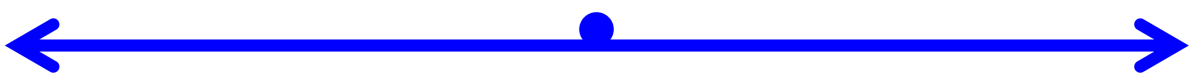
Convergence of a Power Series

There are three possibilities

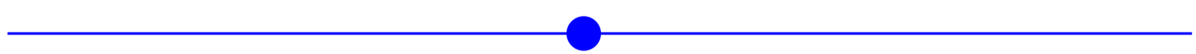
1)There is a positive number R such that the series diverges for $|x-a|>R$ but converges for $|x-a|<R$. The series may or may not converge at the endpoints, $x = a - R$ and $x = a + R$.



2)The series converges for every x. (R = ◎.)



3)The series converges at $x = a$ and diverges elsewhere. $(R = 0)$



What is the interval of convergence?

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + x^4 + \ldots x^n$$

Since r = x, the series converges |x| <1, or -1 < x < 1. In interval notation (-1,1).

Test endpoints of –1 and 1.

$$\sum_{n=0}^{\infty} (-1)^n$$ Series diverges

$$\sum_{n=0}^{\infty} (1)^n$$ Series diverges

Finding interval of convergence

$$\sum_{n=0}^{\infty} \frac{x^n}{n}$$

Use the ratio test:

$$u_n = \frac{x^n}{n} \quad and \quad u_{n+1} = \frac{x^{n+1}}{n+1}$$

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{n+1} \square \frac{n}{x^n}\right| = |x|$$

$|x| < 1$ R=1 (-1, 1)

For x = 1

$$\sum_{n=0}^{\infty} \frac{1}{n}$$

Harmonic series diverges

For x = -1

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$$

Alternating Harmonic series converges

Interval of convergence

[-1, 1)

Differentiation and Integration of Power Series

If the function is given by the series

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots\ldots c_n(x-a)^n \ldots$$

Has a radius of convergence R>0, the on the interval (c-R, c+R) the function is continuous, Differentiable and integrable where:

$$f'(x) = \sum_{n=0}^{\infty} n c_n (x-a)^{n-1}$$

$$\int f(x)dx = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^n}{n+1}$$

The radius of convergence is the same but the interval of convergence may differ at the endpoints.

Constructing Power Series

If a power series exists has a radius of convergence = R

It can be differentiated

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ...c_n(x-a)^n...$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2.....nc_n(x-a)^{n-1}...$$

$$f''(x) = 2c_2 + 2*3c_3(x-a) + 3*4(x-a)^2....$$

$$f'''(x) = 1*2*3c_3 + 2*3*4c_4(x-a) + 3*4*5(x-a)^2 + ...$$

So the nth derivative is

$$f^{(n)}(x) = n!c_n + terms\ \ with\ \ factor\ of\ \ \ (x-a)$$

Finding the coefficients for a Power Series

$$f^{(n)}(x) = n!c_n + terms\ \ with\ \ factor\ of\ \ \ (x-a)$$

All derivatives for f(x) must equal the series Derivatives at x = a.

$$f'(a) = c_1$$

$$f''(a) = 1*2c_2$$

$$f'''(a) = 1*2*3c_3$$

$$f^{(n)}(a) = n!c_n$$

$$\frac{f^{(n)}(a)}{n!} = c_n$$

If f has a series representation centered at x=a, the series must be

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(a)}{k!} = f(a) + f'(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 \ldots$$

$$+\frac{f^{(n)}(a)}{n!}x^n + \ldots$$

If f has a series representation centered at x=0, the series must be

$$\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!} = f(a) + f'(0) + \frac{f''(0)}{2!}(x-a)^2 + \frac{f'''(0)}{3!}(x-a)^3 \ldots$$

$$+\frac{f^{(n)}(0)}{n!}x^n + \ldots$$

Find the derivative and the integral

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \ldots$$