LECTURE NOTES ON FOURIER SERIES

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Introduction: In 1807, Fourier astounded some of his contemporaries by asserting that an "arbitrary" function could be expressed as a linear combination of sines and cosines. These linear combinations, now called **Fourier series,** have become an indispensable tool in the analysis of certain periodical phenomena(such as vibrations, and planetary and wave motion)which are studied in Physics and Engineering.

1.1 DEFINATION: A *trigonometric series* is of the form $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ where the co-efficients a_0, a_n, b_n are constants.

It can be shown that every periodic function of x satisfying certain very general conditions can be represented in the above form.

1.2 DEFINATION: Let $f : [-\pi, \pi] \rightarrow R$ be a bounded function and P = $\{-\pi = t_0 < t_1 < \cdots < t_n = \pi\}$ be a partition of $[-\pi, \pi]$. For any points $x_i \in [t_{i-1}, t_i]$ $(i = 1,2, \dots n)$ $S(P, f) = \sum_{i=1}^{i=n} f(x_i)(t_i - t_{i-1})$ is called Riemann sum of the function f. If $\lim_{||P|| \to 0} S(P, f)$ is finite then f is said to be Riemann integrable on $[-\pi, \pi]$ and the finite limit is denoted by $\int_{-\pi}^{\pi} f$. We call f is integrable on $[-\pi, \pi]$ when it is Riemann integrable on $[-\pi, \pi]$.

1.3 DEFINATION: Let $f : [-\pi, \pi] \rightarrow R$ be unbounded in $[-\pi, \pi]$ and there are finite number of points $-\pi = t_0 < t_1 < \cdots < t_n = \pi$ such that f is bounded and integrable in every closed sub interval contained in each open interval (t_{i-1}, t_i). If $\lim_{\delta_i \to 0} \int_{t_{i-1}+\delta_i}^{t_i - \delta_i} |f|$ exists finitely for each i then f is said to be absolutely convergent in $[-\pi, \pi]$.

1.4 DEFINATION: Let $f : [-\pi, \pi] \rightarrow R$ be an integrable function on $[-\pi, \pi]$ or if unbounded on $[-\pi, \pi]$ let the improper integral $\int_{-\pi}^{\pi} f(x)dx$ be absolutely convergent.Then the trigonometric series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ is called the Fourier series in $[-\pi, \pi]$ corresponding to the function f, where a_0, a_n, b_n, called Fourier co-efficients, are given by $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$ $n \in N$.

1.5 THEOREM: Suppose $f : [-\pi, \pi] \rightarrow R$ be an integrable function on $[-\pi, \pi]$ or if unbounded on $[-\pi, \pi]$ let the improper integral $\int_{-\pi}^{\pi} f(x)dx$ be absolutely convergent. For $n \in N$ let $S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$ be the n th partial sum of the Fourier series and $T_n(x) = \frac{c_0}{2} +$

$\sum_{k=0}^{n}(c_k \cos kx + d_k \sin kx)$ be the n th partial sum of any trigonometric series. Then $\int_{-\pi}^{\pi}(f - S_n)^2 \leq \int_{-\pi}^{\pi}(f - T_n)^2$. Equality holds if and only if $a_k = c_k$ and $b_k = d_k$ for all k.

(That is to say among all functions T_n, S_n will give the best possible mean square approximation to f.)

1.6 Worked out examples:

1.6.1 Find the Fourier series of $f(x) = x, x \in [-\pi, \pi]$

Ans: Since f is continuous on $[-\pi, \pi]$ it is bounded and integrable on $[-\pi, \pi]$. Then the Fourier series of f in $[-\pi, \pi]$ is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$

Where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = 0(\text{since f is odd})$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = 0(\text{since f is odd})$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx\, dx = \frac{2}{\pi}\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} + \frac{1}{n}\int_{0}^{\pi}\cos nx\, dx\right\} = -\frac{2}{n}\cos n\pi$$

$$= \begin{cases} -\frac{2}{n} & \text{if n is even} \\ \frac{2}{n} & \text{if n is odd} \end{cases}$$

Hence the Fourier series for f in $[-\pi, \pi]$ is $2[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots.]$.

1.7 Dirichlet's conditions : A real valued function f in [a, b] is said to satisfy Dirichlet's condition if either

1) f is bounded in [a, b] and piecewise monotone in [a, b] i.e. the interval can be broken up into a finite number of open subintervals in each of which f is monotonic,

Or,2) f has finite number of points of infinite discontinuity in [a, b],but when arbitrary small neighbourhoods of these points of discontinuity are excluded, f(x) is bounded in the remainder of the interval and in each remaining interval f is piecewise monotone and $\int_a^b f(x)dx$ is absolutely convergent.

1.8 THEOREM: If f is a periodic function of period 2π andsatisfies Dirichlet's conditions in $[-\pi, \pi]$, then at x= c the Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ converges to $\frac{1}{2}\{f(c-0) + f(c+0)\}$ for $-\pi < c < \pi$; and to $\frac{1}{2}\{f(\pi - 0) + f(-\pi + 0)\}$ for c = $\pm\pi$.

1.9 Worked out examples:

1.9.1. Obtain the Fourier series expansion of the function f(x) = $x\sin x$ on $[-\pi,\pi]$. Hence deduce that $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$

Ans. Here f is bounded and integrable in $[-\pi,\pi]$. So the Fourier series expansion for f(x) is $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\,dx=\frac{2}{\pi}\{[-x\cos x]_0^{\pi}+\int_0^{\pi}\cos x\,dx\}=\frac{2}{\pi}.\pi=2;$

for $n\in N$, $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{2}{\pi}\int_0^{\pi}x\sin x\cos nx\,dx=\frac{1}{\pi}\int_0^{\pi}x(\sin(n+1)x-\sin(n-1)x)dx=$

$$\begin{cases}\frac{1}{\pi}\left\{\left[-\frac{x\cos 2x}{2}\right]_0^{\pi}+\frac{1}{2}\int_0^{\pi}\cos 2x\,dx\right\}=-\frac{1}{2} \text{ if } n=1\\ \frac{1}{\pi}\left\{\left[\frac{x(.\cos(n-1)x)}{n-1}-x.\frac{\cos(n+1)x}{n+1}\right]_0^{\pi}+\int_0^{\pi}\left[\frac{\cos(n+1)x}{n+1}-\frac{\cos(n-1)x}{n-1}\right]dx\right\}=(-1)^{n-1}\left[\frac{1}{n-1}-\frac{1}{n+1}\right]=\frac{(-1)^{n-1}2}{n^2-1} \text{ if } n\neq 1\end{cases}$$

For $n\in N$, $b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=0$(Since $f(x)\sin nx$ is an odd function).

Hence the Fourier series corresponding to f in $[-\pi,\pi]$ is

$f(x)\sim 1-\frac{1}{2}\cos x+2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos nx$.

Here f is an even function, so f ' is an odd function. Hence f ' is symmetric about origin.

Now $f'(x)=\sin x+x\cos x>0$ *in* $\left(0,\frac{\pi}{2}\right)$ and <0 *in* $(\frac{\pi}{2},\pi)$. Hence f is piecewise monotone in $[-\pi,\pi]$.So f satisfies Dirichlet's conditions in $[-\pi,\pi]$. Since f is continuous at $\frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right)=1-\frac{1}{2}\cos(\frac{\pi}{2})+2\sum_{n=2}^{\infty}\frac{(-1)^{n-1}}{n^2-1}\cos n\left(\frac{\pi}{2}\right).$$

$\therefore \frac{\pi}{2}=1+\frac{2}{2^2-1}-\frac{2}{4^2-1}+\frac{2}{6^2-1}-\cdots$

This implies $\frac{\pi}{4}=\frac{1}{2}+\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$

1.9.2. Find the Fourier series of f where $f(x)=\begin{cases}x-\pi & \text{if } -\pi<x<0\\ \pi-x & \text{if } 0\leq x\leq\pi\end{cases}$

Ans. Here f is bounded and integrable in $[-\pi,\pi]$. So the Fourier series for f(x) is

$\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$ where $a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)dx=\frac{1}{\pi}[\int_{-\pi}^{0}(x-\pi)dx+\int_0^{\pi}(\pi-x)dx]$ $\frac{1}{\pi}\{[\frac{x^2}{2}-\pi x]_{-\pi}^{0}+\left[\pi x-\frac{x^2}{2}\right]_0^{\pi}\}=\frac{1}{\pi}\left(-\frac{3\pi^2}{2}+\frac{\pi^2}{2}\right)=-\pi$

for $n \in N$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}[\int_{-\pi}^{0}(x-\pi)\cos nx\,dx + \int_0^{\pi}(\pi - x)\cos nx\,dx] =$
$\frac{1}{\pi}\{\left[\frac{x-\pi}{n}\sin nx\right]_{-\pi}^{0} + [\frac{\pi-x}{n}\sin nx]_0^{\pi} - \frac{1}{n}[\int_{-\pi}^{0}\sin nx\,dx - \int_0^{\pi}\sin nx\,dx]\} = \frac{1}{n^2\pi}\{[\cos nx]_{-\pi}^{0} - [\cos nx]_0^{\pi}\} =$
$\frac{1}{n^2\pi}\{1-(-1)^n-(-1)^n+1\} = \frac{2}{n^2\pi}(1-(-1)^n)$.

For $n \in N$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}(x-\pi)\sin nx\,dx + \int_0^{\pi}(\pi-x)\sin nx\,dx\right] =$

$\frac{1}{\pi}\{\left[-\frac{x-\pi}{n}\cos nx\right]_{-\pi}^{0} - [\frac{\pi-x}{n}\cos nx]_0^{\pi} + \frac{1}{n}[\int_{-\pi}^{0}\cos nx\,dx - \int_0^{\pi}\cos nx\,dx\} = \frac{2}{n}(1-(-1)^n)$

So the Fourier series of *f*(x) is *f*(x) $\sim -\frac{\pi}{2} + \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ ...\right) + 4\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \cdots\right)$

1.9.3 Obtain the Fourier series expansion of f(x) in $[-\pi, \pi]$ where

f(x) $= \begin{cases} \mathbf{0} & -\pi \le x < 0 \\ \frac{1}{4}\pi x & 0 \le x \le \pi \end{cases}$**. Hence show that $1 + \frac{2}{3^2} + \frac{2}{5^2} + \cdots \quad = \frac{\pi^2}{8}$**

Ans. Hints. $a_0 = \frac{\pi^2}{8}$, $\quad a_n = \frac{1}{4n^2}((-1)^n - 1), b_n = -\frac{\pi}{4n}(-1)^n$

The Fourier series expansion of f(x) in $[-\pi, \pi]$ is $\frac{\pi^2}{16} - \frac{1}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \frac{\pi}{4}\left[\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \cdots\right]$.The function f is bounded and monotonic in $[-\pi, \pi]$.So f satisfies Dirichlet's conditions in $[-\pi, \pi]$. Also f is continuous at 0. So f(0) $= \frac{\pi^2}{16} - \frac{1}{2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$

1.9.4. Find the Fourier series expansion of periodic function f(x) with period 2π defined by

$\mathbf{f(x)} = \begin{cases} \mathbf{0}, & -\pi < x < a\pi \\ \mathbf{1}, & \mathbf{a \le x \le b} \\ \mathbf{0}, & \mathbf{b} < x < \pi \end{cases}$**. Find the sum of the series for $\mathbf{x = 4\pi + a}$ and deduce that**
$\sum_{n=1}^{\infty} \frac{\sin n(b-a)}{n} = \frac{\pi - b + a}{2}$.

Ans. Hints. Define f at $x = \pi$ and at $x = -\pi$ as $f(-\pi) = 0 = f(\pi)$

The function f is bounded and integrable in $[-\pi, \pi]$. The Fourier series of f(x) is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}(b-a)$, $\quad a_n = \frac{1}{n\pi}[\sin nb - \sin na]$, $b_n = \frac{1}{n\pi}[\cos na - \cos nb\,]$. The Fourier series of f(x) is $\frac{1}{2\pi}(b-a) +$
$\frac{1}{\pi}\sum_{n=1}^{\infty}(\frac{1}{n}[(\sin nb - \sin na)\cos nx + (\cos na - \cos nb)\sin nx])$.

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Here f is piecewise monotone and bounded in $[-\pi, \pi]$.. So f satisfies Dirichlet's conditions in $[-\pi, \pi]$

Since f is periodic function of period 2π, so $f(4\pi + a) = f(a)$. Now the Fourier series for f(x) converges at $x = 4\pi + a$ to $\frac{1}{2}[f(a+0) + f(a-0)] = \frac{1}{2}$ That is the sum of the series at $x = 4\pi + a$ is $\frac{1}{2}$

At x=b, the Fourier series converges to $\frac{1}{2}[f(b+0)+f(b-0)]=\frac{1}{2}$. Hence $\frac{1}{2}=\frac{b-a}{2\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n(b-a)}{n}$. So $\sum_{n=1}^{\infty}\frac{\sin n(b-a)}{n}=\frac{\pi-b+a}{2}$.

1.9.5 Let $\mathbf{f}: [-\pi,\pi]\to \mathbf{R}$ be defined as follows: $\mathbf{f(x)}=\begin{cases}-\cos x, & -\pi\le x<0\\ \cos x, & 0\le x\le \pi\end{cases}$. Obtain the Fourier series for the function f(x). Hence find the sum of the series $\frac{2}{1.3}-\frac{6}{5.7}+\frac{10}{9.11}-\cdots$

Ans. Hints. $a_0=0, a_n=0, b_n=\frac{\frac{1}{\pi}4n}{n^2-1}$ if n is even $b_n=0$ if nis odd. At $x=\frac{\pi}{4}$ the sum of the series is $\frac{\pi}{4\sqrt{2}}$

1.9.6. Show that the even function $\mathbf{f(x)}=|\mathbf{x}|$ has a Fourier cosine series in $[-\pi,\pi]$ of the form $\frac{\pi}{2}-\frac{4}{\pi}\left\{\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+\cdots\right\}$.Hence show that $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$.

1.9.7. Prove that the odd function $(\mathbf{x})=\mathbf{e^x-e^{-x}}, \mathbf{x}\in [-\pi,\pi]$ has a Fourier sine series of the form $\frac{4(\sinh\pi)}{\pi}\left[\frac{\sin x}{1^2+1}-\frac{2(\sin 2x)}{2^2+1}+\cdots\right]$. Hence show that $\frac{1}{1^2+1}-\frac{3}{3^2+1}+\frac{5}{5^2+1}+\cdots=\pi/(4\cosh\frac{\pi}{2})$

Fourier sine and cosine series:

If f is a real valued function on $[0,\pi]$ which is bounded and integrable then the series of the form $\sum_{n=1}^{\infty} b_n \sin nx$ is called a Fourier sine seriers, if $b_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$. And the series of the form $\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx$ is called a Fourier cosine seriers, if $a_n=\frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$.

1.9.8. The function f is defined for $0<x<2\pi$ as f(x) =$\begin{cases}(\mathbf{x}-\pi)^2 & \text{when } 0<x<\pi,\\ \pi^2, & \text{when } \pi\le x\le 2\pi.\end{cases}$ Hence deduce the value of $\sum_{n=1}^{\infty}\frac{1}{n^2}$.

Ans. Defining f(0)= π^2, here f is bounded and integrable in $[0,2\pi]$. So the Fourier series for f(x) is $\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$

Where $a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)dx=\frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)^2dx+\int_{\pi}^{2\pi}\pi^2dx\right]=\frac{1}{\pi}\left[\left[\frac{(x-\pi)^3}{3}\right]_0^{\pi}+[\pi^2x]_{\pi}^{2\pi}\right]=\frac{4}{3}\pi^2$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx=\frac{1}{\pi}\left[\int_0^{\pi}(x-\pi)^2\cos nx\, dx+\int_{\pi}^{2\pi}\pi^2\cos nx\, dx\right]$$
$$=\frac{1}{\pi}\{\left[\frac{1}{n}(x-\pi)^2\sin nx\right]_0^{\pi}$$
$$-2/n\int_0^{\pi}(x-\pi)\sin nx\, dx\}=-\frac{2}{n^2\pi}\left\{[-(x-\pi)\cos nx]_0^{\pi}+\int_0^{\pi}\cos nx\, dx\right\}=\frac{2}{n^2}$$

$b_n = \frac{1}{\pi}\{\int_0^{\pi}(x-\pi)^2 \sin nx\, dx + \int_{\pi}^{2\pi}(\pi)^2 \sin nx\, dx\} = \frac{1}{\pi}\Big\{\Big[-\frac{(x-\pi)^2}{n}\cos nx\Big]_0^{\pi} + \frac{2}{n}\int_0^{\pi}(x-\pi)\cos nx\, dx - \frac{\pi^2(\cos nx)}{n}\Big|_{\pi}^{2\pi}\Big\} = \frac{1}{n\pi}\Big\{(-1)^n\pi^2 + \frac{2}{n^2}[\cos nx]_0^{\pi}\Big\} = \frac{1}{\pi}\Big[\frac{(-1)^n\pi^2}{n} - \frac{2}{n^3}(1-(-1)^n)\Big]$. Hence the Fourier series of f(x) in [0, 2π] is $\frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2}{n^2}\cos nx + \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\frac{(-1)^n\pi^2}{n} - \frac{2}{n^3}(1-(-1)^n)\Big]\sin nx$.

Here f is monotonically decreasing in [0, π] and monotonically increasing in [π, 2π](constant function in [π, 2π]). So f satisfies Dirichlet's conditions in [0, 2 π]. At x= 0, the Fourier series converges to ½ [f(0+0)+f(2 π − 0)]. Thus $\pi^2 = \frac{2}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{2}{n^2}$. That is $\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}$.

1.9.9. Find the Fourier cosine series for the function f defined for 0<= x<= π as $\mathbf{f(x)} = \begin{cases} \frac{\pi}{3}, 0 \le x < \frac{\pi}{3} \\ 0, \frac{\pi}{3} < x < \frac{2\pi}{3} \\ -\frac{\pi}{3}, \frac{2\pi}{3} < x \le \pi \end{cases}$ **and** $\mathbf{f\left(\frac{\pi}{3}\right) = \frac{\pi}{12}, f\left(\frac{2\pi}{3}\right) = -\frac{\pi}{12}}$. **Find the sum of the series for x=**$\frac{\pi}{3}$ **and deduce that** $\mathbf{1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \frac{1}{13} - \frac{1}{17} + \cdots = \pi/2\sqrt{3}}$**.**

Ans. Here f is bounded and integrable in [0, π]. So the Fourier cosine series for f is $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$.

Where $a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}\Big\{\int_0^{\frac{\pi}{3}}\frac{\pi}{3}dx + \int_{\frac{2\pi}{3}}^{\pi}\Big(-\frac{\pi}{3}\Big)dx\Big\} = \frac{2}{3}\Big\{\frac{\pi}{3} - \frac{\pi}{3}\Big\} = 0.$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx = \frac{2}{\pi}\Big\{\frac{\pi}{3}\int_0^{\frac{\pi}{3}}\cos nx\, dx - \frac{\pi}{3}\int_{\frac{2\pi}{3}}^{\pi}\cos nx\, dx\Big\} = \frac{2}{3n}\Big\{[\sin nx]_0^{\frac{\pi}{3}} - [\sin nx]_{\frac{2\pi}{3}}^{\pi}\Big\} = \frac{2}{3n}\Big(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\Big) = \frac{4}{3n}\sin\frac{n\pi}{2}\cos\frac{n\pi}{6}$, which is nonzero only when n is odd but not a multiple of 3.

So the Fourier cosine series for the function f(x) is $\frac{4}{3}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi}{6}\cos nx = \frac{2}{\sqrt{3}}\Big[\frac{\cos x}{1} - \frac{\cos 5x}{5} + \frac{\cos 7x}{7} - \frac{\cos 11x}{11} + \cdots\Big]$

Here f is bounded and monotonically decreasing in the interval [0, π]. Thus f satisfies Dirichlet's condition in [0, π]. Also f is continuous at 0. So at x=0, the series converges to f(0). Thus $\frac{\pi}{3} = \frac{2}{\sqrt{3}}\Big(\frac{1}{1} - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \cdots\Big)$. This implies $\frac{\pi}{2\sqrt{3}} = 1 - \frac{1}{5} + \frac{1}{7} - \frac{1}{11} + \cdots$

1.9.10. Prove that for $\mathbf{0 \le x \le \pi, x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right)}$**. Hence deduce that** $\mathbf{x = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right), 0 \le x \le \pi}$**.**

Answer: Suppose $f(x) = x(\pi - x), 0 \le x \le \pi$.

Then the Fourier sine series of f(x) in [0, π] is $\sum_{n=1}^{\infty} b_n \sin nx$, where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx = \frac{2}{\pi}\left\{\left[-\frac{x(\pi-x)}{n}\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos nx\,dx\right\} = \frac{2}{n^2\pi}\{[(\pi - 2x)\sin nx]_0^{\pi} + 2\int_0^{\pi}\sin nx\,dx\} = \frac{4}{n^2\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{4}{n^3\pi}[1-(-1)^n]$.

So the Fourier sine series of f(x) is $\frac{8}{\pi}\left\{\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right\}$.

Here f is continuous in $[0,\pi]$, so it is bounded in $[0,\pi]$. Also f is monotonically increasing in $\left[0,\frac{\pi}{2}\right]$ and monotonically decreasing in $\left[\frac{\pi}{2},\pi\right]$. So f satisfies Dirichlet's conditions in $[0,\pi]$.

Since f is continuous in $[0,\pi]$, $x(\pi - x) = \frac{8}{\pi}\left(\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \cdots\right), 0 \le x \le \pi$.

The last part will be proved later in Chapter SEQUENCE and SERIES of FUNCTIONS.

1.9.11. Find the Fourier series of the periodic function f with period 2π defind by $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ e^x, & 0 < x \le \pi \end{cases}$.

Ans. Here f is bounded and integrable in $[-\pi,\pi]$. So the Fourier series of f(x)is $\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$ where $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} e^x dx = \frac{1}{\pi}(e^{\pi} - 1)$, $a_n = \frac{1}{\pi}\int_0^{\pi} e^x\cos nx\,dx = \frac{1}{\pi}\{[e^x\cos nx]_0^{\pi} + n\int_0^{\pi} e^x\sin nx\,dx\} = \frac{1}{\pi}\{(-1)^n e^{\pi} - 1 - n^2\int_0^{\pi} e^x\cos nx\,dx\}$.

$\therefore\ a_n = \frac{1}{\pi(1+n^2)}\{(-1)^n e^{\pi} - 1\}$.

$b_n = \frac{1}{\pi}\int_0^{\pi} e^x\sin nx\,dx = \frac{1}{\pi}\{[e^x\sin nx]_0^{\pi} - n\int_0^{\pi} e^x\cos nx\,dx\} = -\frac{n}{\pi}\{[e^x\cos nx]_0^{\pi} + n\int_0^{\pi} e^x\sin nx\,dx\} =$

$$-\frac{n}{\pi}\{(-1)^n e^{\pi} - 1 + n\,b_n\}.$$

$\therefore\ b_n = \frac{-n}{\pi(1+n^2)}\{(-1)^n e^{\pi} - 1\}$. So the Fourier series of f in $[-\pi,\pi]$ is $f(x) \sim \frac{1}{2\pi}(e^{\pi} - 1) + \sum_{n=1}^{\infty}\frac{1}{\pi(1+n^2)}\{(-1)^n e^{\pi} - 1\}\cos nx + \sum_{n=1}^{\infty}\frac{-n}{\pi(1+n^2)}\{(-1)^n e^{\pi} - 1\}\sin nx$.

1.9.12. Show that if 0<x<π, $\pi - x = \frac{\pi}{2} + \sum_{n=1}^{\infty}(\sin 2nx)/n$. Show that the equation does not hold for $x = 0$ and $x = \pi$. Explain why it does not hold.

1.10 BESSEL's INEQUALITY: If f : $[-\pi,\pi] \to R$ be continuous except for a finite number of jump discontinuity and is periodic of period 2π then $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$, where a_n and b_n's are Fourier coefficients.

Corollary: If f satisfies the stated conditions, then $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ is convergent.

1.11 PARSEVAL'S IDENTITY: If $\mathbf{a_n, b_n'}$s are Forier coefficients of f in $[-\pi, \pi]$, then $\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$.

1.12.1 Examples : The series $\sum_{n=1}^{\infty} \sin nx$, $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ are not Fourier series of Riemann integrable function in $[-\pi, \pi]$ as $\sum_{n\in N} b_n^2$ is divergent.

1.12.2 Example: Examine whether the trigonometric series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is a Fourier series in $[-\pi, \pi]$.

Answer: Here $a_n = 0$ and $b_n = \frac{1}{n^2}$. Since the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum \frac{1}{n^4}$ is a convergent series so the given series is a Fourier series of some Riemann integrable function in $[-\pi, \pi]$