■ MECHANICS OF A PARTICLE

Let $\mathbf{r}$ be the radius vector of a particle from some given origin and $\mathbf{v}$ its vector velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt}. \tag{1.1}$$

The *linear momentum* $\mathbf{p}$ of the particle is defined as the product of the particle mass and its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1.2}$$

In consequence of interactions with external objects and fields, the particle may experience forces of various types, e.g., gravitational or electrodynamic; the vector sum of these forces exerted on the particle is the total force $\mathbf{F}$. The mechanics of the particle is contained in *Newton's second law of motion*, which states that there exist frames of reference in which the motion of the particle is described by the differential equation

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} \equiv \dot{\mathbf{p}}, \tag{1.3}$$

or

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}). \tag{1.4}$$

In most instances, the mass of the particle is constant and Eq. (1.4) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\mathbf{a}, \tag{1.5}$$

where $\mathbf{a}$ is the vector acceleration of the particle defined by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2}. \tag{1.6}$$

The equation of motion is thus a differential equation of second order, assuming $\mathbf{F}$ does not depend on higher-order derivatives.

A reference frame in which Eq. (1.3) is valid is called an *inertial* or *Galilean system*. Even within classical mechanics the notion of an inertial system is something of an idealization. In practice, however, it is usually feasible to set up a coordinate system that comes as close to the desired properties as may be required. For many purposes, a reference frame fixed in Earth (the "laboratory system") is a sufficient approximation to an inertial system, while for some astronomical purposes it may be necessary to construct an inertial system by reference to distant galaxies.

Many of the important conclusions of mechanics can be expressed in the form of conservation theorems, which indicate under what conditions various mechanical quantities are constant in time. Equation (1.3) directly furnishes the first of these, the

Conservation Theorem for the Linear Momentum of a Particle: If the total force, $\mathbf{F}$*, is zero, then* $\dot{\mathbf{p}} = 0$ *and the linear momentum,* $\mathbf{p}$*, is conserved.*

The angular momentum of the particle about point O, denoted by $\mathbf{L}$, is defined as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{1.7}$$

where $\mathbf{r}$ is the radius vector from O to the particle. Notice that the order of the factors is important. We now define the *moment of force* or *torque* about O as

$$\mathbf{N} = \mathbf{r} \times \mathbf{F}. \tag{1.8}$$

The equation analogous to (1.3) for $\mathbf{N}$ is obtained by forming the cross product of $\mathbf{r}$ with Eq. (1.4):

$$\mathbf{r} \times \mathbf{F} = \mathbf{N} = \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}). \tag{1.9}$$

Equation (1.9) can be written in a different form by using the vector identity:

$$\frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \frac{d}{dt}(m\mathbf{v}). \tag{1.10}$$

where the first term on the right obviously vanishes. In consequence of this iden-

tity, Eq. (1.9) takes the form

$$\mathbf{N} = \frac{d}{dt}(\mathbf{r} \times m\mathbf{v}) = \frac{d\mathbf{L}}{dt} = \dot{\mathbf{L}}. \tag{1.11}$$

Note that both **N** and **L** depend on the point O, about which the moments are taken.

As was the case for Eq. (1.3), the torque equation, (1.11), also yields an immediate conservation theorem, this time the

Conservation Theorem for the Angular Momentum of a Particle: If the total torque, **N**, *is zero then* $\dot{\mathbf{L}} = 0$, *and the angular momentum* **L** *is conserved.*

Next consider the work done by the external force **F** upon the particle in going from point 1 to point 2. By definition, this work is

$$W_{12} = \int_1^2 \mathbf{F} \cdot d\mathbf{s}. \tag{1.12}$$

For constant mass (as will be assumed from now on unless otherwise specified), the integral in Eq. (1.12) reduces to

$$\int \mathbf{F} \cdot d\mathbf{s} = m \int \frac{d\mathbf{v}}{dt} \cdot \mathbf{v}\, dt = \frac{m}{2} \int \frac{d}{dt}(v^2)\, dt,$$

and therefore

$$W_{12} = \frac{m}{2}(v_2^2 - v_1^2). \tag{1.13}$$

The scalar quantity $mv^2/2$ is called the kinetic energy of the particle and is denoted by T, so that the work done is equal to the change in the kinetic energy:

$$W_{12} = T_2 - T_1. \tag{1.14}$$

If the force field is such that the work W_{12} is the same for any physically possible path between points 1 and 2, then the force (and the system) is said to be *conservative*. An alternative description of a conservative system is obtained by imagining the particle being taken from point 1 to point 2 by one possible path and then being returned to point 1 by another path. The independence of W_{12} on

the particular path implies that the work done around such a closed circuit is zero, i.e.:

$$\oint \mathbf{F} \cdot d\mathbf{s} = 0. \tag{1.15}$$

Physically it is clear that a system cannot be conservative if friction or other dissipation forces are present, because $F \cdot d\mathbf{s}$ due to friction is always positive and the integral cannot vanish.

By a well-known theorem of vector analysis, a necessary and sufficient condition that the work, W_{12}, be independent of the physical path taken by the particle is that $\mathbf{F}$ be the gradient of some scalar function of position:

$$\mathbf{F} = -\nabla V(\mathbf{r}), \tag{1.16}$$

where V is called the *potential*, or *potential energy*. The existence of V can be inferred intuitively by a simple argument. If W_{12} is independent of the path of integration between the end points 1 and 2, it should be possible to express W_{12} as the change in a quantity that depends only upon the positions of the end points. This quantity may be designated by $-V$, so that for a differential path length we have the relation

$$\mathbf{F} \cdot d\mathbf{s} = -dV$$

or

$$F_s = -\frac{\partial V}{\partial s},$$

which is equivalent to Eq. (1.16). Note that in Eq. (1.16) we can add to V any quantity constant in space, without affecting the results. Hence *the zero level of V is arbitrary.*

For a conservative system, the work done by the forces is

$$W_{12} = V_1 - V_2. \tag{1.17}$$

Combining Eq. (1.17) with Eq. (1.14), we have the result

$$T_1 + V_1 = T_2 + V_2, \tag{1.18}$$

which states in symbols the

Energy Conservation Theorem for a Particle: If the forces acting on a particle are conservative, then the total energy of the particle, $T - V$, is conserved.

The force applied to a particle may in some circumstances be given by the gradient of a scalar function that depends explicitly on both the position of the particle and the time. However, the work done on the particle when it travels a distance ds,

$$\mathbf{F} \cdot d\mathbf{s} = -\frac{\partial V}{\partial s}\, ds,$$

is then no longer the total change in $-V$ during the displacement, since V also changes explicitly with time as the particle moves. Hence, the work done as the particle goes from point 1 to point 2 is no longer the difference in the function V between those points. While a total energy $T + V$ may still be defined, it is not conserved during the course of the particle's motion.

MECHANICS OF A SYSTEM OF PARTICLES

In generalizing the ideas of the previous section to systems of many particles, we must distinguish between the *external forces* acting on the particles due to sources outside the system, and *internal forces* on, say, some particle i due to all other particles in the system. Thus, the equation of motion (Newton's second law) for the ith particle is written as

$$\sum_j \mathbf{F}_{ji} + \mathbf{F}_i^{(e)} = \dot{\mathbf{p}}_i, \tag{1.19}$$

where $\mathbf{F}_i^{(e)}$ stands for an external force, and $\mathbf{F}_{ji}$ is the internal force on the ith particle due to the jth particle ($\mathbf{F}_{ii}$, naturally, is zero). We shall assume that the $\mathbf{F}_{ij}$ (like the $\mathbf{F}_i^{(e)}$) obey Newton's third law of motion in its original form: that the forces two particles exert on each other are equal and opposite. This assumption (which does not hold for all types of forces) is sometimes referred to as the *weak law of action and reaction*.

Summed over all particles, Eq. (1.19) takes the form

$$\frac{d^2}{dt^2}\sum_i m_i \mathbf{r}_i = \sum_i \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \mathbf{F}_{ji}. \tag{1.20}$$

The first sum on the right is simply the total external force $\mathbf{F}^{(e)}$, while the second term vanishes, since the law of action and reaction states that each pair $\mathbf{F}_{ij} + \mathbf{F}_{ji}$ is zero. To reduce the left-hand side, we define a vector $\mathbf{R}$ as the average of the radii vectors of the particles, weighted in proportion to their mass:

$$\mathbf{R} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i} = \frac{\sum m_i \mathbf{r}_i}{M}. \tag{1.21}$$

The vector $\mathbf{R}$ defines a point known as the *center of mass*, or more loosely as the center of gravity, of the system (cf. Fig. 1.1). With this definition, (1.20) reduces to

$$M\frac{d^2\mathbf{R}}{dt^2} = \sum_i \mathbf{F}_i^{(e)} \equiv \mathbf{F}^{(e)}, \tag{1.22}$$

which states that the center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Purely internal forces, if they obey Newton's third law, therefore have no effect on the

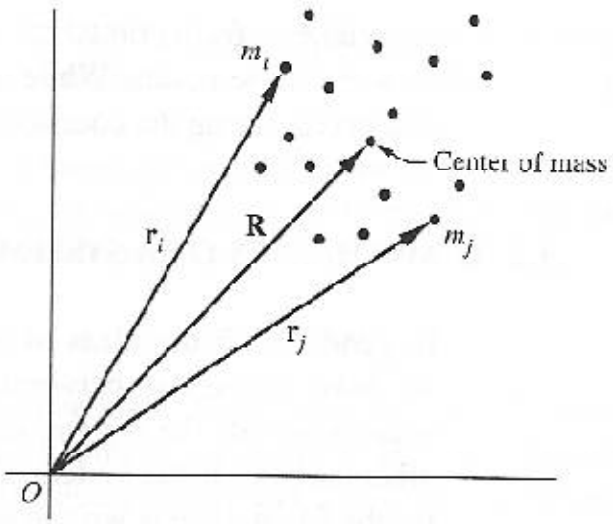


FIGURE 1.1 The center of mass of a system of particles.

motion of the center of mass. An oft-quoted example is the motion of an exploding shell—the center of mass of the fragments traveling as if the shell were still in a single piece (neglecting air resistance). The same principle is involved in jet and rocket propulsion. In order that the motion of the center of mass be unaffected, the ejection of the exhaust gases at high velocity must be counterbalanced by the forward motion of the vehicle at a slower velocity.

By Eq. (1.21) the total linear momentum of the system,

$$\mathbf{P} = \sum m_i \frac{d\mathbf{r}_i}{dt} = M\frac{d\mathbf{R}}{dt}, \tag{1.23}$$

is the total mass of the system times the velocity of the center of mass. Consequently, the equation of motion for the center of mass, (1.23), can be restated as the

Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved.

We obtain the total angular momentum of the system by forming the cross product $\mathbf{r}_i \times \mathbf{p}_i$ and summing over i. If this operation is performed in Eq. (1.19), there results, with the aid of the identity, Eq. (1.10),

$$\sum_i (\mathbf{r}_i \times \dot{\mathbf{p}}_i) = \sum_i \frac{d}{dt}(\mathbf{r}_i \times \mathbf{p}_i) = \dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{(e)} + \sum_{\substack{i,j \\ i \neq j}} \mathbf{r}_i \times \mathbf{F}_{ji}. \quad (1.24)$$

The last term on the right in (1.24) can be considered a sum of the pairs of the form

$$\mathbf{r}_i \times \mathbf{F}_{ji} + \mathbf{r}_j \times \mathbf{F}_{ij} = (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ji}, \quad (1.25)$$

the action and reaction law.* Equations (1.23) and (1.26), and their corresponding conservation theorems, are not applicable in such cases, at least in the form given here. Usually it is then possible to find some generalization of $\mathbf{P}$ or $\mathbf{L}$ that is conserved. Thus, in an isolated system of moving charges it is the sum of the mechanical angular momentum and the electromagnetic "angular momentum" of the field that is conserved.

Equation (1.23) states that the total linear momentum of the system is the same as if the entire mass were concentrated at the center of mass and moving with it. The analogous theorem for angular momentum is more complicated. With the origin O as reference point, the total angular momentum of the system is

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i.$$

Let $\mathbf{R}$ be the radius vector from O to the center of mass, and let $\mathbf{r}_i'$ be the radius vector from the center of mass to the ith particle. Then we have (cf. Fig. 1.3)

$$\mathbf{r}_i = \mathbf{r}_i' + \mathbf{R} \quad (1.27)$$

and

$$\mathbf{v}_i = \mathbf{v}_i' + \mathbf{v}$$

where

$$\mathbf{v} = \frac{d\mathbf{R}}{dt}$$

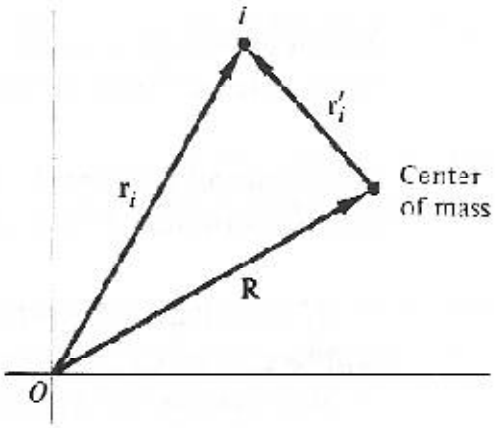


FIGURE 1.3 The vectors involved in the shift of reference point for the angular momentum.

*If two charges are moving uniformly with parallel velocity vectors that are not perpendicular to the line joining the charges, then the net mutual forces are equal and opposite but do not lie along the vector between the charges. Consider, further, two charges moving (instantaneously) so as to "cross the T," i.e., one charge moving directly at the other, which in turn is moving at right angles to the first. Then the second charge exerts a nonvanishing magnetic force on the first, without experiencing any magnetic reaction force at that instant.

is the velocity of the center of mass relative to O, and

$$\mathbf{v}'_i = \frac{d\mathbf{r}'_i}{dt}$$

is the velocity of the ith particle relative to the center of mass of the system. Using Eq. (1.27), the total angular momentum takes on the form

$$\mathbf{L} = \sum_i \mathbf{R} \times m_i \mathbf{v} + \sum_i \mathbf{r}'_i \times m_i \mathbf{v}'_i + \left(\sum_i m_i \mathbf{r}'_i \right) \times \mathbf{v} + \mathbf{R} \times \frac{d}{dt} \sum_i m_i \mathbf{r}'_i .$$

The last two terms in this expression vanish, for both contain the factor $\sum m_i \mathbf{r}'_i$, which, it will be recognized, defines the radius vector of the center of mass in the very coordinate system whose origin is the center of mass and is therefore a null vector. Rewriting the remaining terms, the total angular momentum about O is

$$\mathbf{L} = \mathbf{R} \times M\mathbf{v} + \sum_i \mathbf{r}'_i \times \mathbf{p}'_i . \tag{1.28}$$

In words, Eq. (1.28) says that the total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass, plus the angular momentum of motion about the center of mass. The form of Eq. (1.28) emphasizes that in general $\mathbf{L}$ depends on the origin O, through the vector $\mathbf{R}$. Only if the center of mass is at rest with respect to O will the angular momentum be independent of the point of reference. In this case, the first term in (1.28) vanishes, and $\mathbf{L}$ always reduces to the angular momentum taken about the center of mass.

■ CONSTRAINTS

From the previous sections one might obtain the impression that all problems in mechanics have been reduced to solving the set of differential equations (1.19):

$$m_i \ddot{\mathbf{r}}_i = \mathbf{F}_i^{(e)} + \sum_j \mathbf{F}_{ji}.$$

One merely substitutes the various forces acting upon the particles of the system, turns the mathematical crank, and grinds out the answers! Even from a purely physical standpoint, however, this view is oversimplified. For example, it may be necessary to take into account the *constraints* that limit the motion of the system. We have already met one type of system involving constraints, namely rigid bodies, where the constraints on the motions of the particles keep the distances r_{ij} unchanged. Other examples of constrained systems can easily be furnished. The beads of an abacus are constrained to one-dimensional motion by the supporting wires. Gas molecules within a container are constrained by the walls of the vessel to move only *inside* the container. A particle placed on the surface of a solid sphere is subject to the constraint that it can move only on the surface or in the region exterior to the sphere.

Constraints may be classified in various ways, and we shall use the following system. If the conditions of constraint can be expressed as equations connecting the coordinates of the particles (and possibly the time) having the form

$$f(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots, t) = 0, \tag{1.37}$$

then the constraints are said to be *holonomic*. Perhaps the simplest example of holonomic constraints is the rigid body, where the constraints are expressed by equations of the form

$$(\mathbf{r}_i - \mathbf{r}_j)^2 - c_{ij}^2 = 0.$$

A particle constrained to move along any curve or on a given surface is another obvious example of a holonomic constraint, with the equations defining the curve or surface acting as the equations of a constraint.

Constraints not expressible in this fashion are called nonholonomic. The walls of a gas container constitute a nonholonomic constraint. The constraint involved in the example of a particle placed on the surface of a sphere is also nonholonomic, for it can be expressed as an inequality

$$r^2 - a^2 \geq 0$$

(where a is the radius of the sphere), which is not in the form of (1.37). Thus, in a gravitational field a particle placed on the top of the sphere will slide down the surface part of the way but will eventually fall off.

Constraints are further classified according to whether the equations of constraint contain the time as an explicit variable (rheonomous) or are not explicitly dependent on time (scleronomous). A bead sliding on a rigid curved wire fixed in space is obviously subject to a scleronomous constraint; if the wire is moving in some prescribed fashion, the constraint is rheonomous. Note that if the wire moves, say, as a reaction to the bead's motion, then the time dependence of the constraint enters in the equation of the constraint only through the coordinates of the curved wire (which are now part of the system coordinates). The overall constraint is then scleronomous.

Constraints introduce two types of difficulties in the solution of mechanical problems. First, the coordinates r_i are no longer all independent, since they are connected by the equations of constraint; hence the equations of motion (1.19) are not all independent. Second, the forces of constraint, e.g., the force that the wire exerts on the bead (or the wall on the gas particle), is not furnished a priori. They are among the unknowns of the problem and must be obtained from the solution we seek. Indeed, imposing constraints on the system is simply another method of stating that there are forces present in the problem that cannot be specified directly but are known rather in terms of their effect on the motion of the system.

In the case of holonomic constraints, the first difficulty is solved by the introduction of *generalized coordinates*. So far we have been thinking implicitly in terms of Cartesian coordinates. A system of N particles, free from constraints, has $3N$ independent coordinates or *degrees of freedom*. If there exist holonomic constraints, expressed in k equations in the form (1.37), then we may use these equations to eliminate k of the $3N$ coordinates, and we are left with $3N - k$ independent coordinates, and the system is said to have $3N - k$ degrees of freedom.

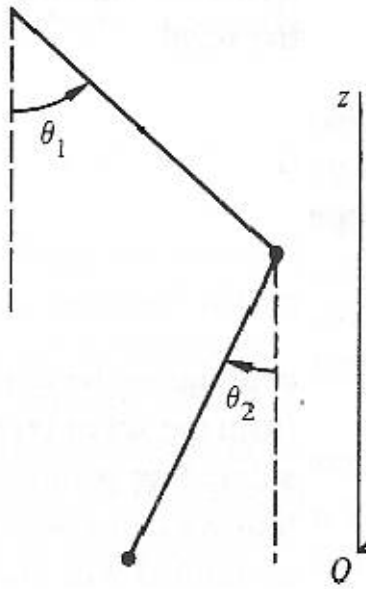
This elimination of the dependent coordinates can be expressed in another way, by the introduction of new, $3N - k$, independent variables $q_1, q_2, \ldots, q_{3N-k}$ in terms of which the old coordinates $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ are expressed by equations of the form

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1(q_1, q_2, \ldots, q_{3N-k}, t) \\ &\vdots \\ \mathbf{r}_N &= \mathbf{r}_N(q_1, q_2, \ldots, q_{3N-k}, t) \end{aligned} \tag{1.38}$$

containing the constraints in them implicitly. These are *transformation* equations from the set of $(\mathbf{r}_l)$ variables to the (q_l) set, or alternatively Eqs. (1.38) can be considered as parametric representations of the $(\mathbf{r}_l)$ variables. It is always assumed that we can also transform back from the (q_l) to the $(\mathbf{r}_l)$ set, i.e., that Eqs. (1.38) combined with the k equations of constraint can be inverted to obtain any q_i as a function of the $(\mathbf{r}_l)$ variable and time.

Usually the generalized coordinates, q_l, unlike the Cartesian coordinates, will not divide into convenient groups of three that can be associated together to form vectors. Thus, in the case of a particle constrained to move *on* the surface of a sphere, the two angles expressing position on the sphere, say latitude and longitude, are obvious possible generalized coordinates. Or, in the example of a double pendulum moving in a plane (two particles connected by an inextensible light rod and suspended by a similar rod fastened to one of the particles), satisfactory generalized coordinates are the two angles θ_1, θ_2. (Cf. Fig. 1.4.) Generalized coordinates, in the sense of coordinates other than Cartesian, are often useful in systems without constraints. Thus, in the problem of a particle moving in an external central force field ($V = V(r)$), there is no constraint involved, but it is clearly more convenient to use spherical polar coordinates than Cartesian coordinates. Do not, however, think of generalized coordinates in terms of conventional orthogonal position coordinates. All sorts of quantities may be impressed to serve as generalized coordinates. Thus, the amplitudes in a Fourier expansion of $\mathbf{r}_j$ may be used as generalized coordinates, or we may find it convenient to employ quantities with the dimensions of energy or angular momentum.

If the constraint is nonholonomic, the equations expressing the constraint cannot be used to eliminate the dependent coordinates. An oft-quoted example of a nonholonomic constraint is that of an object rolling on a rough surface without slipping. The coordinates used to describe the system will generally involve angular coordinates to specify the orientation of the body, plus a set of coordinates describing the location of the point of contact on the surface. The constraint of "rolling" connects these two sets of coordinates; they are not independent. A change in the position of the point of contact inevitably means a change in its orientation. Yet we cannot reduce the number of coordinates, for the "rolling" condition is not expressible as a equation between the coordinates, in the manner of (1.37). Rather, it is a condition on the *velocities* (i.e., the point of contact is stationary), a differential condition that can be given in an integrated form only *after* the problem is solved.



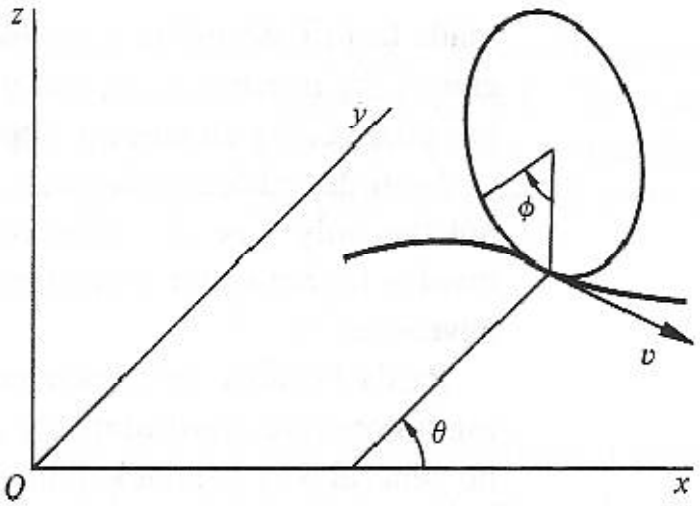


FIGURE 1.4 Double pendulum. **FIGURE 1.5** Vertical disk rolling on a horizontal plane.

A simple case will illustrate the point. Consider a disk rolling on the horizontal xy plane constrained to move so that the plane of the disk is always vertical. The coordinates used to describe the motion might be the x, y coordinates of the center of the disk, an angle of rotation ϕ about the axis of the disk, and an angle θ between the axis of the disk and say, the x axis (cf. Fig 1.5). As a result of the

constraint the velocity of the center of the disk, **v**, has a magnitude proportional to $\dot{\phi}$,

$$v = a\dot{\phi},$$

where a is the radius of the disk, and its direction is perpendicular to the axis of the disk:

$$\dot{x} = v \sin\theta$$
$$\dot{y} = -v \cos\theta.$$

Combining these conditions, we have two *differential* equations of constraint:

$$\begin{aligned} dx - a \sin\theta d\phi - 0, \\ dy + a \cos\theta d\phi = 0. \end{aligned} \tag{1.39}$$

Neither of Eqs. (1.39) can be integrated without in fact solving the problem; i.e., we cannot find an integrating factor $f(x, y, \theta, \phi)$ that will turn either of the equations into perfect differentials (cf. Derivation 4).* Hence, the constraints cannot be reduced to the form of Eq. (1.37) and are therefore nonholonomic. Physically we can see that there can be no direct functional relation between ϕ and the other coordinates x, y, and θ by noting that at any point on its path the disk can be made to roll around in a circle tangent to the path and of arbitrary radius. At the end of the process, x, y, and θ have been returned to their original values, but ϕ has changed by an amount depending on the radius of the circle.

Nonintegrable *differential* constraints of the form of Eqs. (1.39) are of course not the only type of nonholonomic constraints. The constraint conditions may involve higher-order derivatives, or may appear in the form of inequalities, as we have seen.

*In principle, an integrating factor can always be found for a first-order differential equation of constraint in systems involving only two coordinates and such constraints are therefore holonomic. A familiar example is the two-dimensional motion of a circle rolling on an inclined plane.

Partly because the dependent coordinates can be eliminated, problems involving holonomic constraints are always amenable to a formal solution. But there is no general way to attack nonholonomic examples. True, if the constraint is nonintegrable, the differential equations of constraint can be introduced into the problem along with the differential equations of motion, and the dependent equations eliminated, in effect, by the method of Lagrange multipliers.

We shall return to this method at a later point. However, the more vicious cases of nonholonomic constraint must be tackled individually, and consequently in the development of the more formal aspects of classical mechanics, it is almost invariably assumed that any constraint, if present, is holonomic. This restriction does not greatly limit the applicability of the theory, despite the fact that many of the constraints encountered in everyday life are nonholonomic. The reason is that the entire concept of constraints imposed in the system through the medium of wires or surfaces or walls is particularly appropriate only in macroscopic or large-scale problems. But today physicists are more interested in atomic and nuclear problems. On this scale all objects, both in and out of the system, consist alike of molecules, atoms, or smaller particles, exerting definite forces, and the notion of constraint becomes artificial and rarely appears. Constraints are then used only as mathematical idealizations to the actual physical case or as classical approximations to a quantum-mechanical property, e.g., rigid body rotations for "spin." Such constraints are always holonomic and fit smoothly into the framework of the theory.

To surmount the second difficulty, namely, that the forces of constraint are unknown a priori, we should like to so formulate the mechanics that the forces of constraint disappear. We need then deal only with the known applied forces. A hint as to the procedure to be followed is provided by the fact that in a particular system with constraints, i.e., a rigid body, the work done by internal forces (which are here the forces of constraint) vanishes. We shall follow up this clue in the ensuing sections and generalize the ideas contained in it.

■ D'ALEMBERT'S PRINCIPLE AND LAGRANGE'S EQUATIONS

A virtual (infinitesimal) displacement of a system refers to a change in the configuration of the system as the result of any arbitrary infinitesimal change of the coordinates δr_i, *consistent with the forces and constraints imposed on the system at the given instant* t. The displacement is called virtual to distinguish it from an actual displacement of the system occurring in a time interval dt, during which the forces and constraints may be changing. Suppose the system is in equilibrium; i.e., the total force on each particle vanishes, $\mathbf{F}_i = 0$. Then clearly the dot product $\mathbf{F}_i \cdot \delta\mathbf{r}_i$, which is the virtual work of the force $\mathbf{F}_i$ in the displacement $\delta\mathbf{r}_i$, also vanishes. The sum of these vanishing products over all particles must likewise be zero:

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.40}$$

As yet nothing has been said that has any new physical content. Decompose $\mathbf{F}_i$ into the applied force, $\mathbf{F}_i^{(a)}$, and the force of constraint, $\mathbf{f}_i$,

$$\mathbf{F}_i = \mathbf{F}_i^{(a)} + \mathbf{f}_i, \tag{1.41}$$

so that Eq. (1.40) becomes

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta\mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta\mathbf{r}_i = 0. \tag{1.42}$$

We now restrict ourselves to systems for which *the net virtual work of the forces of constraint is zero*. We have seen that this condition holds true for rigid bodies and it is valid for a large number of other constraints. Thus, if a particle is constrained to move on a surface, the force of constraint is perpendicular to the surface, while the virtual displacement must be tangent to it, and hence the virtual work vanishes. This is no longer true if sliding friction forces are present, and we must exclude such systems from our formulation. The restriction is not unduly hampering, since the friction is essentially a macroscopic phenomenon. On the other hand, the forces of rolling friction do not violate this condition, since the forces act on a point that is momentarily at rest and can do no work in an infinitesimal displacement consistent with the rolling constraint. Note that if a particle is

constrained to a surface that is itself moving in time, the force of constraint is instantaneously perpendicular to the surface and the work during a virtual displacement is still zero even though the work during an actual displacement in the time dt does not necessarily vanish.

We therefore have as the condition for equilibrium of a system that the virtual work of the *applied forces* vanishes:

$$\sum_i \mathbf{F}_i^{(a)} \cdot \delta \mathbf{r}_i = 0. \tag{1.43}$$

Equation (1.43) is often called the *principle of virtual work*. Note that the coefficients of $\delta \mathbf{r}_i$ can no longer be set equal to zero; i.e., in general $\mathbf{F}_i^{(a)} \neq 0$, since the $\delta \mathbf{r}_i$ are not completely independent but are connected by the constraints. In order to equate the coefficients to zero, we must transform the principle into a form involving the virtual displacements of the q_i, which are independent. Equation (1.43) satisfies our needs in that it does not contain the $\mathbf{f}_i$, but it deals only with statics; we want a condition involving the general motion of the system.

To obtain such a principle, we use a device first thought of by James Bernoulli and developed by D'Alembert. The equation of motion,

$$\mathbf{F}_i = \dot{\mathbf{p}}_i,$$

can be written as

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0,$$

which states that the particles in the system will be in equilibrium under a force equal to the actual force plus a "reversed effective force" $-\dot{\mathbf{p}}_i$. Instead of (1.40), we can immediately write

$$\sum_i (\mathbf{F}_i - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \tag{1.44}$$

and, making the same resolution into applied forces and forces of constraint, there results

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i + \sum_i \mathbf{f}_i \cdot \delta \mathbf{r}_i = 0.$$

We again restrict ourselves to systems for which the virtual work of the forces of constraint vanishes and therefore obtain

$$\sum_i (\mathbf{F}_i^{(a)} - \dot{\mathbf{p}}_i) \cdot \delta \mathbf{r}_i = 0, \tag{1.45}$$

which is often called *D'Alembert's principle*. We have achieved our aim, in that the forces of constraint no longer appear, and the superscript $^{(a)}$ can now be dropped without ambiguity. It is still not in a useful form to furnish equations of motion for the system. We must now transform the principle into an expression involving virtual displacements of the generalized coordinates, which are then independent of each other (for holonomic constraints), so that the coefficients of the δq_i can be set separately equal to zero.

The translation from $\mathbf{r}_i$ to q_j language starts from the transformation equations (1.38),

$$\mathbf{r}_i = \mathbf{r}_i(q_1, q_2, \ldots, q_n, t) \tag{1.45'}$$

(assuming n independent coordinates), and is carried out by means of the usual "chain rules" of the calculus of partial differentiation. Thus, $\mathbf{v}_i$ is expressed in terms of the $\dot{q}_k$ by the formula

$$\mathbf{v}_i \equiv \frac{d\mathbf{r}_i}{dt} = \sum_k \frac{\partial \mathbf{r}_i}{\partial q_k} \dot{q}_k + \frac{\partial \mathbf{r}_i}{\partial t}. \tag{1.46}$$

Similarly, the arbitrary virtual displacement $\delta \mathbf{r}_i$ can be connected with the virtual displacements δq_i by

$$\delta \mathbf{r}_i = \sum_j \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j. \tag{1.47}$$

Note that no variation of time, δt, is involved here, since a virtual displacement by definition considers only displacements of the coordinates. (Only then is the virtual displacement perpendicular to the force of constraint if the constraint itself is changing in time.)

In terms of the generalized coordinates, the virtual work of the $\mathbf{F}_i$ becomes

$$\sum_i \mathbf{F}_i \cdot \delta \mathbf{r}_i = \sum_{i,j} \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j$$
$$= \sum_j Q_j \delta q_j, \tag{1.48}$$

where the Q_j are called the components of the *generalized force*, defined as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1.49}$$

Note that just as the q's need not have the dimensions of length, so the Q's do not necessarily have the dimensions of force, but $Q_j \delta q_j$ must always have the dimensions of work. For example, Q_j might be a torque N_j and dq_j a differential angle $d\theta_j$, which makes $N_j\, d\theta_j$ a differential of work.

We turn now to the other other term involved in Eq. (1.45), which may be written as

$$\sum_i \dot{\mathbf{p}}_i \cdot \delta \mathbf{r}_i = \sum_i m_i \ddot{\mathbf{r}}_i \cdot \delta \mathbf{r}_i.$$

Expressing $\delta \mathbf{r}_i$ by (1.47), this becomes

$$\sum_{i,j} m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \delta q_j.$$

Consider now the relation

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt} \left(m_i \dot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} \right) - m_i \dot{\mathbf{r}}_i \cdot \frac{d}{dt} \left(\frac{\partial \mathbf{r}_i}{\partial q_j} \right) \right]. \tag{1.50}$$

In the last term of Eq. (1.50) we can interchange the differentiation with respect to t and q_j, for, in analogy to (1.46),

$$\frac{d}{dt}\left(\frac{\partial \mathbf{r}_i}{\partial q_j}\right) - \frac{\partial \dot{\mathbf{r}}_i}{\partial q_j} = \sum_k \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial q_k}\dot{q}_k + \frac{\partial^2 \mathbf{r}_i}{\partial q_j \partial t},$$

$$= \frac{\partial \mathbf{v}_i}{\partial q_j},$$

by Eq. (1.46). Further, we also see from Eq. (1.46) that

$$\frac{\partial \mathbf{v}_i}{\partial \dot{q}_j} = \frac{\partial \mathbf{r}_i}{\partial q_j}. \tag{1.51}$$

Substitution of these changes in (1.50) leads to the result that

$$\sum_i m_i \ddot{\mathbf{r}}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = \sum_i \left[\frac{d}{dt}\left(m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}_j}\right) - m_i \mathbf{v}_i \cdot \frac{\partial \mathbf{v}_i}{\partial q_j}\right],$$

and the second term on the left-hand side of Eq. (1.45) can be expanded into

$$\sum_j \left\{\frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right)\right] - \frac{\partial}{\partial q_j}\left(\sum_i \frac{1}{2} m_i v_i^2\right) - Q_j\right\} \delta q_j.$$

Identifying $\sum_i \frac{1}{2} m_i v_i^2$ with the system kinetic energy T, D'Alembert's principle (cf. Eq. (1.45)) becomes

$$\sum \left\{\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right] - Q_j\right\} \delta q_j = 0. \tag{1.52}$$

Note that in a system of Cartesian coordinates the partial derivative of T with respect to q_j vanishes. Thus, speaking in the language of differential geometry, this term arises from the curvature of the coordinates q_j. In polar coordinates, e.g., it is in the partial derivative of T with respect to an angle coordinate that the centripetal acceleration term appears.

Thus far, no restriction has been made on the nature of the constraints other than that they be workless in a virtual displacement. The variables q_j can be any set of coordinates used to describe the motion of the system. If, however, the constraints are holonomic, then it is possible to find sets of independent coordinates q_j that contain the constraint conditions implicitly in the transformation equations

(1.38). Any virtual displacement δq_j is then independent of δq_k, and therefore the only way for (1.52) to hold is for the individual coefficients to vanish:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j. \tag{1.53}$$

There are n such equations in all.

When the forces are derivable from a scalar potential function V,

$$\mathbf{F}_i = -\nabla_i V.$$

Then the generalized forces can be written as

$$Q_j = \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{r}_i}{\partial q_j} = -\sum_i \nabla_i V \cdot \frac{\partial \mathbf{r}_i}{\partial q_j},$$

which is exactly the same expression for the partial derivative of a function $-V(\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N, t)$ with respect to q_j:

$$Q_j = -\frac{\partial V}{\partial q_i}. \tag{1.54}$$

Equations (1.53) can then be rewritten as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_i} = 0. \tag{1.55}$$

The equations of motion in the form (1.55) are not necessarily restricted to conservative systems; only if V is not an explicit function of time is the system conservative (cf. p. 4). As here defined, the potential V does not depend on the generalized velocities. Hence, we can include a term in V in the partial derivative with respect to $\dot{q}_j$:

$$\frac{d}{dt}\left(\frac{\partial (T - V)}{\partial \dot{q}_j}\right) - \frac{\partial (T - V)}{\partial q_j} = 0.$$

Or, defining a new function, the *Lagrangian* L, as

$$L = T - V, \tag{1.56}$$

the Eqs. (1.53) become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0, \tag{1.57}$$

expressions referred to as "Lagrange's equations."

Note that for a particular set of equations of motion there is no unique choice of Lagrangian such that Eqs. (1.57) lead to the equations of motion in the given generalized coordinates. Thus, in Derivations 8 and 10 it is shown that if $L(q, \dot{q}, t)$ is an approximate Lagrangian and $F(q, t)$ is *any* differentiable function of the generalized coordinates and time, then

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{dF}{dt} \tag{1.57'}$$

is a Lagrangian also resulting in the same equations of motion. It is also often possible to find alternative Lagrangians beside those constructed by this prescription (see Exercise 20). While Eq. (1.56) is always a suitable way to construct a Lagrangian for a conservative system, it does not provide the *only* Lagrangian suitable for the given system.